5-6 TWO-STREAM INSTABILITY; LINEAR ANALYSIS

The model consists of two opposing streams of charged particles as sketched in Figure 5-6a. Models with relative motion between two sets or streams of charged particles have been studied in great detail since papers by Haeff (1949) and Pierce (1948). Detailed knowledge of the nonlinear behavior of opposing streams came much later, from the simulations done by Dawson (1962). The fluid analog was given much earlier, as by H. Hertz in the 1880's; see comprehensive books on hydrodynamics and acoustics, such as Lamb (1945) or Rayleigh (1945).

One can readily see that an opposing stream system is unstable. When two streams move through each other one wavelength in one cycle of the plasma frequency, a density perturbation on one stream is reinforced by the forces due to bunching of particles in the other stream and vice versa; hence

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![Diagram](image)

**Figure 5-6a** (a) Two opposing streams as seen in the laboratory. (b) The streams in phase space at the start of the problem, \( t = 0 \). (c) The streams in velocity space at \( t = 0 \) and \( t > 0 \).
$\Delta n_1 \propto n_1$, so that the perturbation grows exponentially in time. This simple relation was put forth in 1948 by Professor M. Chodorow of Stanford [and buried in Birdsall's dissertation (Birdsall, 1951)] for two streams moving in the same direction (Chodorow and Suskind, 1964). The phase relation for reinforcement is written as

$$
(v_{\text{relative}}) \left( \frac{2\pi}{\omega_p} \right) = \frac{2\pi}{k}
$$

which for $v_{\text{relative}} = v_0 - (-v_0) = 2v_0$ is

$$
k = \frac{\omega_p}{2v_0}
$$

This $k$ is very close to that found from analysis for maximum growth rate.

The longitudinal linear dielectric function for two independent cold streams may be obtained as was done in Section 5-3 by applying the equations of motion and continuity separately for each stream and adding the currents of each in the field equation. The result is

$$
\frac{1}{\varepsilon_0} \epsilon(\omega, k) = 1 - \frac{\omega_{p1}^2}{(\omega - k \cdot v_{01})^2} - \frac{\omega_{p2}^2}{(\omega - k \cdot v_{02})^2}
$$

for two streams with drift velocities $v_{01}$ and $v_{02}$. This result is also obtainable directly from the usual Vlasov-Poisson set by letting the velocity distribution be two delta functions,

$$
f_0(v) = A\delta(v - v_{01}) + B\delta(v - v_{02})
$$

A system of $N$ independent cold streams produces a sum over streams or species $s$:

$$
\frac{1}{\varepsilon_0} \epsilon(\omega, k) = 1 - \sum_{s=1}^{N} \frac{\omega_{ps}^2}{(\omega - k \cdot v_{0s})^2}
$$

[Extension of the sum to an integral, for $N \to \infty$, must be done carefully, both analytically as shown by Dawson (1960), and also in simulation when a discrete set of beams is used to approximate a smooth distribution $f(v)$ as shown by Biers (1970), and Gitomer and Adam (1976), and discussed in Chapter 16.]

The solutions for complex $\omega$, assuming real $k$ (i.e., an absolute instability, growth in time only, no convection in space), opposing streams of equal strength, $\omega_{p1} = \omega_{p2} \equiv \omega_p$, $v_{01} = v_{02} \equiv v_0$, is found from $\epsilon(\omega, k) = 0$ which is quartic in $\omega$ with four independent solutions. These are

$$
\omega = \pm \left[ k^2 v_0^2 + \omega_p^2 \pm \omega_p \left( 4k^2 v_0^2 + \omega_p^2 \right)^{1/2} \right]^{1/2}
$$

for which
\[
0 < \frac{k \nu_0}{\omega_p} < \sqrt{2} \quad \begin{cases} 
\text{two roots are real} \\
\text{two roots are imaginary}
\end{cases} 
\quad (7)
\]

\[
\sqrt{2} < \frac{k \nu_0}{\omega_p} \quad \text{all four roots are real} 
\quad (8)
\]

\[
\frac{k \nu_0}{\omega_p} = \frac{\sqrt{3}}{2}, \quad \omega_{\text{imaginary}} = \frac{\omega_p}{2}, \quad \text{maximum growth rate} 
\quad (9)
\]

This behavior is sketched in Figure 5-6b; the growth (\(\omega_{\text{imaginary}}\)) is given in more detail in Figure 5-6c.

In this model, where there is growth (\(\omega_{\text{imaginary}} > 0\)), we find that \(\omega_{\text{real}} = 0\); that is, there is no oscillatory part associated with the growth, a situation which is not generally true.

A point of Figure 5-6c is to make clear the existence of a minimum unstable length L of the system; in this model (normalized)

\[
\frac{\omega_p L}{v_0} > \frac{2\pi}{\sqrt{2}} 
\quad \text{(unstable)} 
\quad (10)
\]

in order to obtain growth. This is the same as (7) using \(L = 2\pi / k_0\), where \(k_0\) is the smallest wavenumber in the system.

Growth which begins at small amplitude continues until the streaming is destroyed; indeed, the distribution becomes nearly Maxwellian. Hence, we say that "the colliding streams have thermalized," although not by collisions.

\[\text{Figure 5-6b Dispersion, or } \omega-k, \text{ diagram for two equal opposing streams, real } k, \text{ complex } \omega.\]

The uncoupled space-charge waves are shown dashed. For each value of k, there are four values of \(\omega\) that correspond to four linearly independent waves.
Instead, collective effects build up large electric fields at long wavelengths ($\lambda >>$ particle spacing) and these scatter the particles in phase space.

As the instability grows, two changes are readily observed in $f(v)$ as indicated for $t > 0$ in Figure 5-6a(c). The width of each beam increases [measured directly on an $f(v)$ plot or by $(v^2 - \bar{v}^2)$ of one stream], which is taken as an increase in the temperature of each beam (but perhaps carelessly so, for if the electric field were suddenly shut off—and you should try this—the spread might decrease). The drift or mean velocity $\bar{v}$ decreases.

We might expect, as $v_{\text{thermal}}$ increases and $v_{\text{drift}}$ decreases, that the conditions for linear growth would cease to be met [see Stringer (1964), who shows the threshold for growth for electron-electron streams to be $v_{\text{drift}} \approx 1.3 v_{\text{thermal}}$] and that the exponential growth would stop. However, at this time, the conditions for linearity are largely violated, with perturbed charge densities comparable to the zero-order density; particles in one stream are about to pass their neighbors and wrap into vortices in phase space, that is, become trapped. Hence, the growth need not stop, although we might be tempted to look for a change in character of the growth (e.g., away from exponential in time) at the time where $v_0$ exceeds $\bar{v}/1.3$; keep this in mind in your project. Of course, ES1 can readily be run with warm beams; hence, look for growth with $v_0 = 2\bar{v}$ (Section 5-9), but stability with $v_0 = \bar{v}$.