Homework problems on geometric integration – PiTP 2009

1. A particle orbiting in a spherically symmetric potential conserves its angular momentum per unit mass $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ and therefore remains on a surface of constant angular momentum in phase space. Do the modified Euler and leapfrog integrators conserve this geometric property? Does the Runge-Kutta integrator?

2. Write code to follow the motion of a test particle orbiting a point mass M, with semi-major axis a and eccentricity e, using two different integrators: fourth-order Runge-Kutta and leapfrog. You may assume that the motion is in the x-y plane and that the orbit starts from apocenter, so the initial conditions are

$$x = a(1+e), \quad y = 0, \quad v_x = 0, \quad v_y = \left[\frac{GM(1-e)}{a(1+e)}\right]^{1/2}.$$

The motion is to be followed for N orbital periods, where the period is $2\pi(a^3/GM)^{1/2}$. The energy and angular momentum are

$$E = \tfrac{1}{2}(v_x^2 + v_y^2) - \frac{GM}{r}, \quad L = xv_y - yv_x.$$

The output should contain the maximum fractional error in energy and angular momentum, $|\Delta E_{\max}/E|$ and $|\Delta L_{\max}/L|$ and the total number of force evaluations.

The results should be independent of the parameters G, M, and a so you can choose these to be unity. For $N = 10^4$, plot $|\Delta E_{\max}/E|$ and $|\Delta L_{\max}/L|$ as a function of the number of force evaluations per orbit for both Runge-Kutta and leapfrog. Do this for eccentricity $e = 0.5$, 0.9, 0.99 and 0.999. Compare the performance of the two integrators as a function of eccentricity.

3. Using the MERCURY software package (`http://www.arm.ac.uk/ jec/home.html`) integrate the orbits of the four outer planets (Jupiter, Saturn, Uranus, Neptune) and Pluto from the present time, for 10^6 years into the future. Plot the positions of the planets at 100 yr intervals in two ways: (a) in an inertial frame centered on the Sun; (b) in a frame centered on the Sun that rotates with Neptune. The curious shape of Pluto's orbit in (b) arises because it is in a 3:2 resonance with Neptune.

Some hints on using MERCURY: The package offers several integration methods, only two of which are symplectic: `mvs` (the Wisdom-Holman integrator) and `hybrid` (Wisdom-Holman except during close encounters, but there should be no close encounters during this integration). For these integrators a timestep of 40 days should work fine. You may use the initial conditions in the file big.in after editing out the inner planets, and you will want to remove the comet initial conditions in small.in. Note that the files *.out, *.dmp, and *.tmp must be removed before starting a new run.

4. We argued in the lecture that leapfrog with fixed timestep h is symplectic, but leapfrog with a timestep that depends on phase-space position, $h(\mathbf{r}, \mathbf{v})$, is generally not symplectic. (a) Suppose that we run leapfrog with a variable timestep that is specified in advance, i.e. the n^{th} timestep h_n is taken from a table prepared in advance. Show that this method is symplectic. (b) Suppose that we run leapfrog once with a timestep $h_n = h(\mathbf{x}, \mathbf{v})$ that is determined by the phase-space position, store the set of h_n in a table, and then use these to run leapfrog again from the same initial conditions. According to the arguments above, the first run is not symplectic but the second one is; but the output of the two integrations is identical. What is the resolution to this apparent paradox?