PiTP 2009: Computational Astrophysics

Computational Methods for Numerical Relativity

Lecture 2: The ADM 3+1 decomposition

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Outline

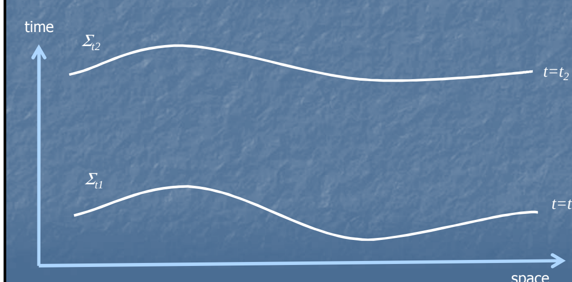
- The ADM (Arnowitt Deser Misner-1962) decomposition
  - A "natural" way to separate 4D spacetime into 3D space + 1D time, which is essential for posing the field equations as in initial boundary value (Cauchy) problem
    - cleanly (at least much as is possible in GR) highlights the separation between the constraints, evolution and gauge degrees of freedom in the metric
    - is the starting point for the York-Lichnerowitz procedure for solving the initial data problem
      - this is used to provide consistent initial data with harmonic evolution
    - is the starting point for the BSSN evolution system
- An example in spherical symmetry: the Einstein-Klein-Gordon system of equations

The ADM decomposition

- There are two logically distinct steps in performing a 3+1 decomposition of the field equations
  - (1) The differential geometry "machinery" of how to describe a d-dimensional subspace in terms of its embedding in a larger (d+1)-dimensional space
  - (2) Applying this decomposition to the Einstein equations
- Note: the exact form of the equations that these days are referred to as the "ADM" equations are not quite in the form presented by ADM, rather this is a reformulation due to J.W. York (in *Sources of Gravitational Radiation*, 1978)

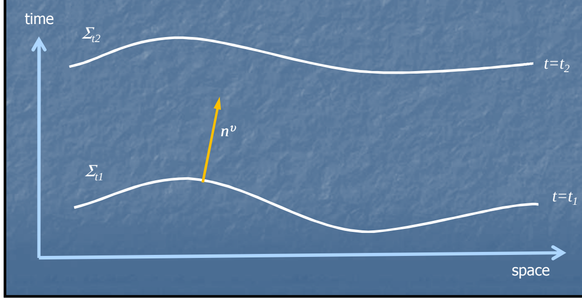
3+1 splitting of spacetime

- Begin by foliating spacetime into a sequence of spacelike hypersurfaces Σ_t, labelled by a monotonic time scalar function t



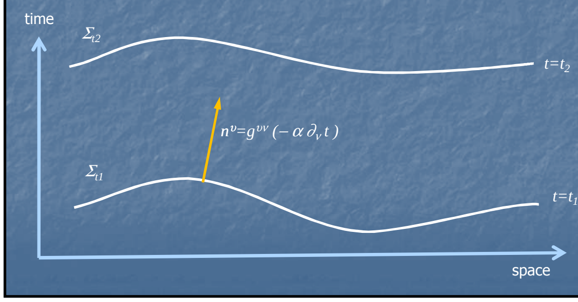
3+1 splitting of spacetime

- Let n^ν denote the *unit timelike* ($n^\nu n_\nu = -1$) vector field normal to Σ



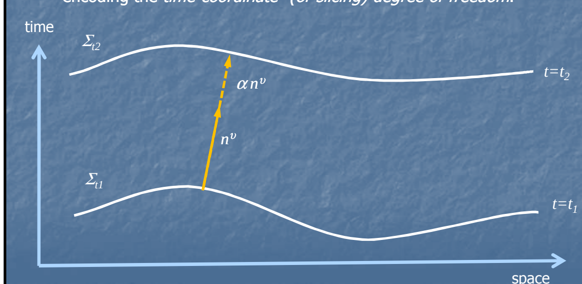
3+1 splitting of spacetime

- The one-form n_ν must be proportional to (minus) the gradient of t : $n_\nu = -\alpha\,\partial_\nu t$. The (positive) function of proportionality α is called the *lapse function.*



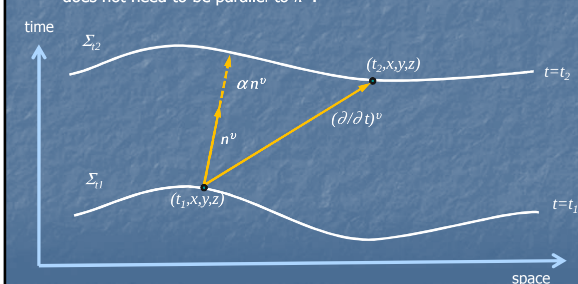
3+1 splitting of spacetime

- The *proper time* interval $\Delta\tau$ that an observer moving along n^ν from t_1 to t_2 will measure is just $\alpha\,(t_2 - t_1)$. Since the labeling of the surfaces by the time function t is arbitrary, one can think of α as encoding the *time-coordinate (or slicing) degree of freedom.*



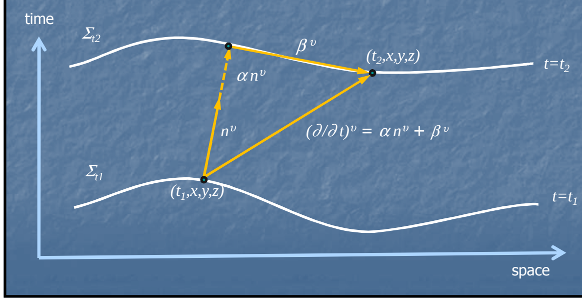
3+1 splitting of spacetime

- We are further free to choose any spatial coordinate system on the hypersurface; in particular the spatial coordinates labeling a point do *not* need to flow along n^ν ; i.e, the *time flow vector field* $(\partial/\partial t)^\nu$ does not need to be parallel to n^ν :



3+1 splitting of spacetime

- The spatial vector β^υ that describes how the intrinsic coordinates flow from one time slice relative to the normal is called the *shift vector*; the shift vector thus encodes the spatial coordinate degrees of freedom in the metric.



3+1 splitting of spacetime

- The geometry intrinsic to a spatial hypersurface Σ is given by a metric h_{ij}. The relationship between the spatial metric and spacetime metric is

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

or, in terms of the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\alpha^2 dt^2 + h_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

- h_{ij} also serves as *a projection tensor*, projecting 4 dimensional tensorial objects onto the 3 dimensional hypersurface; namely

$$\perp T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m} = T^{i_1' \ldots i_n'}{}_{j_1' \ldots j_m'} h_{i_1'}{}^{i_1} \ldots h_{i_n'}{}^{i_n} h^{j_1'}{}_{j_1} \ldots h^{j_m'}{}_{j_m}$$

3+1 splitting of spacetime

- A tensorial object is considered *spatial* if contraction of any one of its indices with the normal vector is zero. E.g. the shift vector is spatial, as is the spatial metric itself:

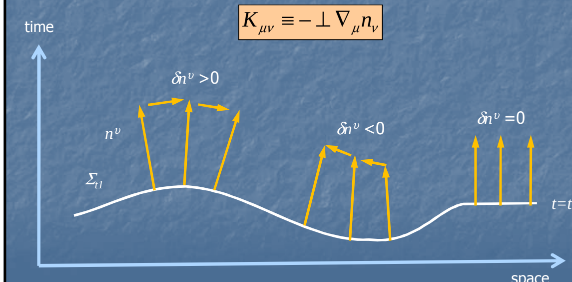
$$\beta^\mu n_\mu = 0$$
$$h^{\mu\nu} n_\mu = h^{\mu\nu} n_\nu = 0$$

- In the end, we would like to have a set of spatial tensors that completely characterizes Σ, and then from the Einstein equations obtain the constraints in terms of these objects, and evolution equations that govern how they evolve in time
  - the spatial metric by itself does not by itself provide all this information; in addition we need to now how Σ is embedded in the higher dimensional space. The *extrinsic curvature tensor* K_{ij}, *also called the second fundamental form*, provides this information.

3+1 splitting of spacetime

- The extrinsic curvature tensor K_{ij} is defined as follows, with a sign convention that gives concave surfaces positive curvature

$$K_{\mu\nu} \equiv -\perp \nabla_\mu n_\nu$$



3+1 splitting of spacetime

- Note that by definition the extrinsic curvature is spatial; a couple of other important properties include:

$$K_{\mu\nu} = K_{\nu u}$$

and

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n h_{\mu\nu}$$

where $\mathcal{L}$ is the *Lie derivative* : $\mathcal{L}_{\mathbf{n}}\mathbf{T}$ represents the change of a tensorial object **T** advected along a vector field **n**.

- The extrinsic curvature is therefore essentially the "velocity" of the spatial metric along the normal direction.

3+1 splitting of spacetime

- However, we are interested in integrating the equations with time "t"
- Using $(\partial/\partial t)^\nu = \alpha n^\nu + \beta^\nu$ and a linearity property of the Lie derivative, we can recast the definition of the extrinsic curvature into the following form:

$$\partial_t h_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta h_{\mu\nu}$$

- we now have all the elements to completely describe a 4D geometry in terms of purely spatial geometric objects, namely h_{ij} and K_{ij} , and the gauge functions α and β^i
- next, we want to see what the Einstein equations look like terms of these quantities

Einstein equations in 3+1 form

- The derivation involves computing all projections of the Einstein equations normal and onto the hypersurface Σ, and using various identities from differential geometry relating projected 4 dimensional to intrinsic 3 dimensional curvature tensors (in particular the Gauss-Codazzi and Gauss-Weingarten relations).
    - will only give the final results … many good references available, in particular MTW (1973), and York's 1978 article in Sources of Gravitational radiation.
- First, we need to define the projected stress-energy tensor components: the energy density ρ, the momentum flux j^i, and the spatial stress tensor S^{ij} :

$$\rho \equiv T_{ab} n^a n^b$$
$$j^i \equiv -\perp\left(T^{ib} n_b\right)$$
$$S^{ij} \equiv \perp T^{ij}$$

Einstein equations in 3+1 form

- The n^μ-n^μ projection of the EFE gives the *Hamiltonian or energy constraint equation*

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

- The prefix (3) notation denotes an intrinsic 3-dimensional object, in this case the Ricci scalar of Σ. K is the trace of the extrinsic curvature tensor.
- The n^μ-$h^{\mu\nu}$ projection gives the 3 *momentum constraint equations*

$$D_b K^{ab} - D^a K = 8\pi j^a$$

where $D_b = \perp\nabla_b$ is spatial covariant derivative operator

- Notice that these 4 equations *only* include spatial objects; i.e. they are 4 *constraints* amongst the 12 unique components of h_{ij} and K_{ij} , imposed by the Einstein equations
    - solving the constraints is a non-trivial exercise, in particular since there is no unique way to define what the freely specifiable vs. constrained degrees of freedom in h_{ij} and K_{ij} are

Einstein equations in 3+1 form

- The $h^{\mu\nu}$ projection of the EFE gives evolution equations for the extrinsic curvature, and this, together with the definition of K_{ij}, form the *evolution equations* :

$$\partial_t K_{ab} = \mathcal{L}_\beta K_{ab} - D_a D_b \alpha + \alpha\left[{}^{(3)}R_{ab} + K\,K_{ab} - 2K_{ad}K^d_b - 8\pi\left(S_{ab} - \frac{h_{ab}}{2}(S-\rho)\right)\right]$$

$$\partial_t h_{ab} = \mathcal{L}_\beta h_{ab} - 2\alpha K_{ab}$$

- One can show that beginning with initial data that satisfies the constraints, a specification of the gauge in terms α and β^ν, imposition of consistent boundary conditions, and coupling to matter that conserves (convariant) energy and momentum, a solution of the evolution equations will satisfy the constraint equations for all time (will demonstrate later with generalized harmonic evolution)

Einstein equations in 3+1 form

- Summary
  - we have fixed the character of the equations by demanding that 1 coordinate be timelike and hypersurface orthogonal, the remaining 3 to be spacelike, factoring out the coordinate degrees of freedom, and choosing the fundamental variables that will be evolved to be purely spatial tensors
  - obtained a coupled hyperbolic/elliptic (*if* some further manipulations of the constraints are made) system, however the hyperbolic subsystem is generically only *weakly hyperbolic,* which can pose problems for numerical integration
  - furthermore, in 3D scenarios, small violations of the constraints (sourced by truncation error, for example) tend to grow exponentially with time, making the ADM form, as is, not too useful for generic evolutions
  - however, in certain symmetry reduced situations, with appropriate gauges, it is a perfectly adequate numerical evolution scheme
    - will look at a spherically symmetric example, which will also be instructive to show that even though the lapse and shift are usually considered "gauge", the coupled nature of the equations don't always allow a clean distinction

Gravity in Spherical Symmetry

- General relativity does not allow any monopole radiation, and thus there are no gravitational waves in spherical symmetry. Thus, any dynamics in a spherically symmetric scenario is entirely driven by matter.
- As a natural extension to project 1, we will use a massless scalar field $\Phi(r,t)$ as the matter source; i.e., now the scalar field back-reacts, and will in fact be the full source of non-trivial geometry
  - its equation of motion remains the same

$$\nabla^\alpha \nabla_\alpha \phi = 0$$

  - and its stress-energy tensor is

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\gamma \Phi \nabla^\gamma \Phi$$

Isotropic coordinates

- We will use a different system of coordinates than before, namely *maximal isotropic coordinates.*
- In isotropic coordinates one takes the spatial metric to be *conformally flat,* which we can always do in spherical symmetry (not so in general)

$${}^{(3)}ds^2 = h_{ij}dx^i dx^j = \psi^4\left[\eta_{ij}dx^i dx\right] = \psi^4\left[dr^2 + r^2 d\Omega^2\right]$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

- The full 4-metric we can thus be written as

$$ds^2 = \left(-\alpha^2 + \psi^4\beta^2\right)dt^2 + 2\psi^4\beta dr dt + \psi^4\left[dr^2 + r^2 d\Omega^2\right]$$

with all metric functions α, β and ψ depending on r and t.

Maximal slicing

- The *maximal slicing* gauge condition is obtained by demanding that the trace of the extrinsic curvature $K=0$ for all time
  - this condition seems "strange" at a first glance, as we just showed that the evolution of K_{ij} is governed by the Einstein equations, so how can we *choose* that its trace be 0?
  - answer is that the trace of $\partial K_{ij}/\partial t$ contains the Laplacian of the lapse, so by choosing the lapse to satisfy that equation with K and $\partial K/\partial t$ set to zero, we find a time slicing (if it exists) where $K=0$:
- With the maximal and isotropic conditions we've exhausted all gauge freedom, though we've done so in a non-traditional manor; i.e. placed conditions on the spatial metric and extrinsic curvature, rather than the lapse and shift
  - in spherical symmetry there are only 2 independent components to K_{ij} ; with the choice $K=0$ we've eliminated 1, and it will turn out to be more economical in the equations to simply substitute in the definition K_{ij} in terms of the metric
  - this will also put the momentum constraint equation into the form of an elliptic equation for the shift vector, clearly showing that we don't have any more coordinate freedom left

Constraint and Slicing Equations

- first, as before, we introduce the following variables for the gradients of the scalar field (excuse the slight change in notation!)

$$\xi(r,t) \equiv \partial_r \Phi; \quad \Pi(r,t) \equiv \psi^2/\alpha\left(\partial_t \Phi - \beta\partial_r \Phi\right)$$

- The Hamiltonian constraint is

$$\psi'' + \frac{2\psi'}{r} + \frac{\psi^5}{12}\left[\frac{1}{\alpha}\left(\beta' - \frac{\beta}{r}\right)\right]^2 + \pi\psi\left[\xi^2 + \Pi^2\right] = 0$$

- The momentum constraint is

$$\beta'' + \left(\beta' - \frac{\beta}{r}\right)\left[\frac{2}{r} + \frac{6\psi'}{\psi} - \frac{\alpha'}{\alpha}\right] + \frac{12\pi\alpha\xi\Pi}{\psi^2} = 0$$

- The maximal slicing condition is

$$\alpha'' + \alpha'\left[\frac{2}{r} + \frac{2\psi'}{\psi}\right] - \frac{2\psi^4}{3\alpha}\left(\beta' - \frac{\beta}{r}\right)^2 - 8\pi\alpha\Pi^2 = 0$$

Evolution equations

- $K=0$ gives the only metric evolution equation :

$$\dot{\psi} - \beta\left(\frac{\psi}{3r} + \psi'\right) - \frac{\psi\beta'}{6} = 0$$

- The scalar field evolution equations are

$$\dot{\Pi} - \frac{1}{r^2\psi^4}\left[r^2\psi^4\left(\beta\Pi + \frac{\alpha\xi}{\psi^2}\right)\right]' + \frac{2}{3}\Pi\left[\beta' + \frac{2\beta}{r}\left(1 + \frac{3r\psi'}{\psi}\right)\right] = 0$$

$$\dot{\xi} - \left(\frac{\alpha\Pi}{\psi^2} + \beta\xi\right)' = 0$$

Evolution strategy

- Notice that we have two independent equations for ψ … therefore have two choices for how to evolve the equations
  - fully constrained evolution : solve all three elliptics at each time step (explicitly shows that there is no gravitational dynamics in spherical symmetry … all the dynamics comes from the coupling to the scalar field)
  - partially constrained evolution : solve the elliptics for the lapse and shift, but evolve the conformal factor with the hyperbolic equation
- Will only get the same, consistent solution to the field equations in a partially constrained evolution *if*
  - initial conditions for ψ are supplied by a solving the Hamiltonian constraint
  - consistent *boundary conditions* are employed … this is not easy to do, and in practice there will often be an O(1/R) level inconsistency, where R is the location of the outer boundary (so choose R sufficiently large so that this error is small)
- During a fully constrained evolution if a black hole forms and excision is employed, will only get a consistent solution if the evolution equation for ψ is used to set the inner boundary condition

Boundary conditions

- Beginning from smooth initial data, i.e. no black holes, there are two classes of boundary conditions
  - (a) Outer *boundary conditions* at r= R
    - demand that the metric is asymptotically flat

$$\psi(r=R,t)=1+\frac{a_o(t)}{r}+O\left(\frac{1}{r^2}\right)$$
$$\alpha(r=R,t)=1+\frac{b_o(t)}{r}+O\left(\frac{1}{r^2}\right)$$
$$\beta(r=R,t)=\frac{c_o(t)}{r}+O\left(\frac{1}{r^2}\right)$$

    - and as in project 1, place (approximate) outgoing radiation (Sommerfeld) conditions on the scalar field

Boundary conditions

- Beginning from smooth initial data, i.e. no black holes, there are two classes of boundary conditions
  - (b) *Regularity* at r= 0
    - the metric is singular at the origin, however this is just an artifact of having chosen spherical polar-type coordinates
    - therefore, demand that solutions be regular at r=0; i.e., this is *not* a boundary condition in the traditional sense, as we're solving the equations within a spherical-like volume of radius R, and r=R is the only boundary in the physical problem
    - here (and this is quite typical for regularity conditions) all regular fields will have *either* an even or odd power series expansion about r=0 of the form

$$f_e(r=0,t)=a_o(t)+a_2(t)r^2+a_4(t)r^4+\ldots$$
$$f_o(r=0,t)=b_1(t)r+b_3(t)r^3+\ldots$$

    - thus, for even functions place a Neumann condition, for odd functions a Dirichlet condition
  - Here, α, Π and ψ have Neumann conditions, β and ξ Dirichlet

Project 2

- A handout describing the spherically symmetric Einstein Klein Gordon equations is available on the web
- Interested students can go through this from the beginning, though that would be a rather lengthy endeavor
- Alternatively, a working RNPL + Fortran code (to handle the elliptics) is given, up to "problem 5". The suggestion would then be to continue with problems 6 & 7:
  - study black hole formation
  - explore the early stages of critical phenomena at the threshold of black hole formation
- Next lecture : adaptive mesh refinement, parallelization, the AMRD/PAMR software packages, critical phenomena example