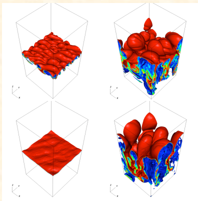
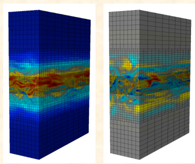
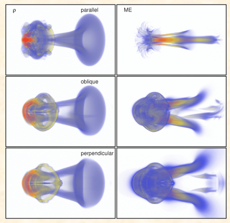
Grid-based methods for hydrodynamics,
MHD, and radiation hydrodynamics.

(Four lectures)







Jim Stone
Department of Astrophysical Sciences
Princeton University

Outline of lectures

Lecture 1. Introduction to physics and numerics

Lecture 2. Operator split (ZEUS-like) methods

Lecture 3. Godunov (PPM-like) methods

Lecture 4. Radiation Hydrodynamics

Lecture 1:
Introduction to physics and numerics.

1. Why fluid dynamics?
2. Equations of MHD.
3. Waves, shocks, & instabilities.
4. Numerical analysis of hyperbolic PDEs.
5. Some basic difference methods.

The Fluid Universe

Most of the "big" questions in astrophysics require studying the fluid dynamics of the visible matter.

- How do galaxies form?
- How do stars form?
- How do planets form?

This requires solving the equations of radiation magneto-hydrodynamics (MHD).

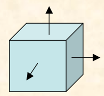
MHD equations: conservation laws

…for mass, momentum, energy, and magnetic flux.

Mass conservation:

Rate of change of mass in a volume is divergence of fluxes through surface

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$



ρ = mass density

v = velocity

$\frac{\partial}{\partial t}$ = Eulerian derivative (at a fixed point in space)

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ = Lagrangian derivative (moving with flow)

Momentum conservation:

Rate of change of momentum within a volume is divergence of stress on surface of volume (no viscous stress)

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} + P^*] = 0$$

Energy conservation:

Rate of change of total energy density E is equal to the divergence of energy flux through the surface

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = 0$$

$E = \rho v^2/2 + e + B^2/2$ is total energy

$P^* = P + B^2/2$ is total pressure (gas + magnetic)

Flux conservation:

Given by Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$ (constraint rather than evolutionary equation)

From Ohm's Law, the current and electric field are related by

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$$

For a fully conducting plasma, $\sigma \longrightarrow \infty$

So c**E** = -(**v** x **B**).

$$\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

The results are the equations of compressible inviscid ideal MHD:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} + P^*] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = 0$$

$$\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

Where $E = \rho v^2/2 + e + B^2/2$ is total energy

$P^* = P + B^2/2$ is total pressure (gas + magnetic)

Plus an equation of state $P = P(\rho, T)$

Warning: used units so that μ=1

Can also be written in non-conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \rho \mathbf{v}\mathbf{v} = -\nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot e\mathbf{v} = -\frac{p}{\rho} \nabla \cdot \mathbf{v}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Plus an equation of state $P = P(\rho, T)$

Useful form for numerical methods based on operator splitting (lecture 2)

Equation of state

Usually adopt the ideal gas law $P = nkT$

In thermal equilibrium, each internal degree of freedom has energy (kT/2). Thus, internal energy density for an ideal gas with *m* internal degrees of freedom

$$e = nm(kT/2).$$

Combining, $P = (\gamma - 1)e$ where $\gamma = (m+2)/m$

For monoatomic gas (H), γ=5/3 (*m*=3)

diatomic gas (H_2), γ=7/5 (*m*=5)

Also common to use isothermal EOS $P = C^2\rho$ where C=isothermal sound speed when (radiative cooling time) << (dynamical time)

In some circumstances, an ideal gas law is not appropriate, and must use more complex (or tabular) EOS (e.g. for degenerate matter)

Hyperbolic conservation laws.

Hydrodynamic equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] = 0,$$
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + \mathrm{P}] = 0,$$
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = 0,$$

Can all be written in a compact form (in 1D):

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \quad \text{where} \quad \mathbf{U} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ E \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + P \\ \rho v_x v_y \\ \rho v_x v_z \\ (E+P)v_x \end{bmatrix},$$

Rewrite as: $\boxed{\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial x} = 0}$ a hyperbolic PDE.

If $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$ = constant, then we have a linear, hyperbolic PDE.

There is much analysis on solution of hyperbolic PDEs.

(…actually, MHD equations are not strictly hyperbolic)

Advection (entropy wave)

If P and V_x are constant, it is easy to find time-dependent solutions to the hydro equations representing advection (entropy wave).

Recall: $\mathbf{U} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ E \end{bmatrix}$ $\mathbf{F} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + P \\ \rho v_x v_y \\ \rho v_x v_z \\ (E+P)v_x \end{bmatrix}$, so $\frac{\partial \mathbf{F}}{\partial \mathbf{U}} = V_x$

Fluid equations become $\frac{\partial \mathbf{U}}{\partial t} + v_x \frac{\partial \mathbf{U}}{\partial x} = 0$

Which has solution: $f(x,t) = f(x - v_x t, 0)$

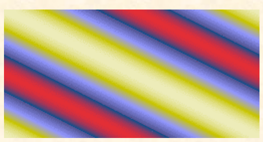
At any later time, solution is just initial condition displaced by $v_x t$. In particular, density field moves with flow without changing shape (advection). Even discontinuous solutions for density are allowed, and just move with flow (contact discontinuties).

Sound waves

Another important characteristic of hyperbolic PDEs is they admit solutions of the form:

$$a = a_0 + a_1 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) \quad \text{(WAVES)}$$

When a_1/a_0 << 1; waves are small amplitude; linear
When a_1/a_0 > 1, waves are large amplitude, nonlinear (in this case, plane wave solution does not persist, for example nonlinear terms cause steepening)



Movie of density in linear sound wave

Linear waves are produced by small amplitude disturbances, with v < C (sound waves)

Dispersion relation for *hydrodynamic* waves.

Substitute solution for plane waves into hydrodynamic equations. Assume a uniform homogeneous background medium, so a_0=constant, and v_0=0. Keep only linear terms. Fluid equations become:

$$-i\omega\rho_1 = -i\rho_0 \mathbf{k}\cdot\mathbf{v_1}$$
$$-i\omega\mathbf{v_1} = -i\frac{1}{\rho_0}\mathbf{k}P_1$$
$$-i\omega P_1 = -i\gamma P_0 \mathbf{k}\cdot\mathbf{v_1}$$

Linear system with constant coefficients! Solutions require det(A)=0, which requires

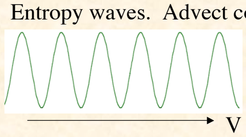
$\boxed{\omega^3(\omega^2 - C^2k^2) = 0}$ where $C^2 = \gamma P_0/\rho_0$ is the adiabatic sound speed

Apparently 5 modes; 3 advection modes and 2 sound waves with

$$\omega/k = \pm C$$

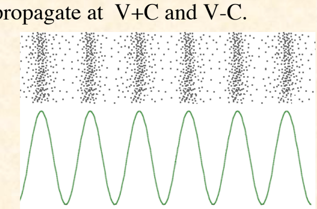
Summary of wave modes in *hydrodynamics*:

1. Entropy waves. Advect constant density field at V.



e.g. advection of sinusoidal density profile

2. Sound waves. Density, velocity, and pressure fluctuations that propagate at V+C and V-C.



Dispersion relation for MHD waves.

Substitute solution for plane waves into MHD equations. Assume a uniform homogeneous background medium, so a_0=constant, and v_0=0. Get a much more complicated dispersion relation (derivation is non-trivial! see Jackson):

$$[\omega^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2][\omega^4 - \omega^2 k^2(v_A^2 + C^2) + k^2C^2(\mathbf{k}\cdot\mathbf{v}_A)^2] = 0$$

Where $\mathbf{v}_A = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}$ is the Alfven speed

$C^2 = \gamma P_0/\rho_0$ is the sound speed

There are three modes (only one in hydrodynamics!):

Alfven wave propagates at V_A

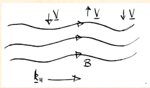
Slow and fast magnetosonic waves propagating at C_s and C_f

(Of course, the entropy mode is also present in both cases)

MHD Wave Modes.

1. **Alfven Waves**

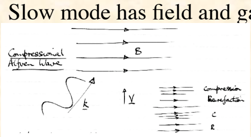
   Zero-frequency when k perpendicular to B (propagate along B), incompressible. Represent propagating transverse perturbations of field.



$$\mathbf{V}_A = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}$$

2. **Fast and Slow Magnetosonic Waves**

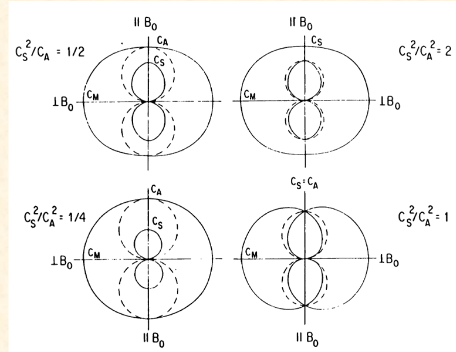
   Compressible perturbations of both field and gas.
   Fast mode has field and gas compression in phase
   Slow mode has field and gas compression out of phase.



$$C^2_{f,s} = \frac{1}{2}\left([C^2 + C_A^2] \pm \sqrt{[C^2 + C_A^2]^2 - 4C^2C^2_{Ax}}\right)$$

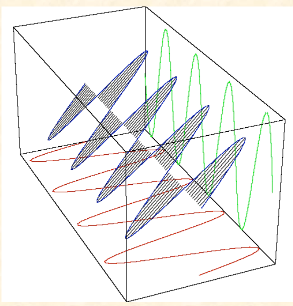
$$C_A^2 = (B_x^2 + B_y^2 + B_z^2)/i(4\pi\rho), \qquad C^2_{Ax} = B_x^2/(4\pi\rho)$$

Phase velocities of MHD waves: Friedrichs diagrams.



Note for in some cases, modes are degenerate. Eigenvalues of linearized MHD equations are not always linearly independent. MHD equations are not *strictly hyperbolic*.

Circularly polarized Alfven waves.



Since Alfven waves require transverse velocities, they can be polarized.

Circularly polarized waves are the sum of two linear polarizations with fixed phase shift.

Even in 1D, MHD requires 2 transverse velocity components in order to capture all MHD wave modes!

Wave Steepening

So far, we've only considered linear solutions.
Most important term we have ignored is the $(\mathbf{v} \cdot \nabla)\mathbf{v}$ term in the momentum equation.

This term produces wave-steepening.

(Recall solutions to Burger's equation, $\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x} = 0$)

We can estimate for the magnitude of this term (how long does a linear wave take to steepen into a discontinuity -- a shock?):

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \sim k\mathbf{v}_1^2 \quad \text{while} \quad \frac{\partial \mathbf{v}}{\partial t} \sim \omega\mathbf{v}_1$$

The two are comparable after N ~ (ω/kv_1) wave periods

Shocks

Remarkably, hyperbolic PDEs admit discontinuous solutions.

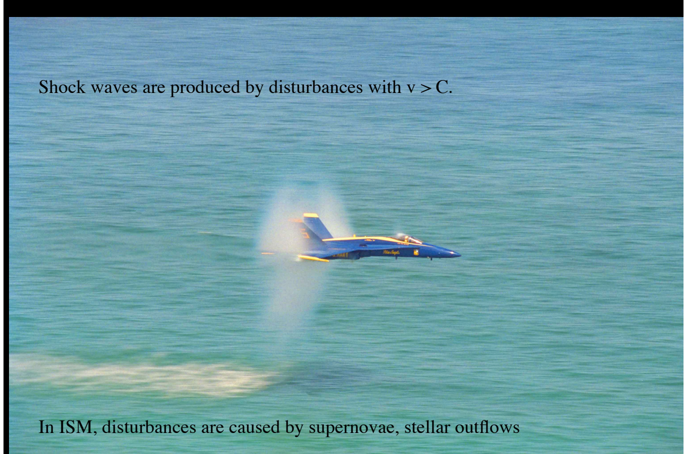
For the equations of MHD, the simplest example is a contact discontinuity: discontinuous change in density (with constant P) advected at constant v.

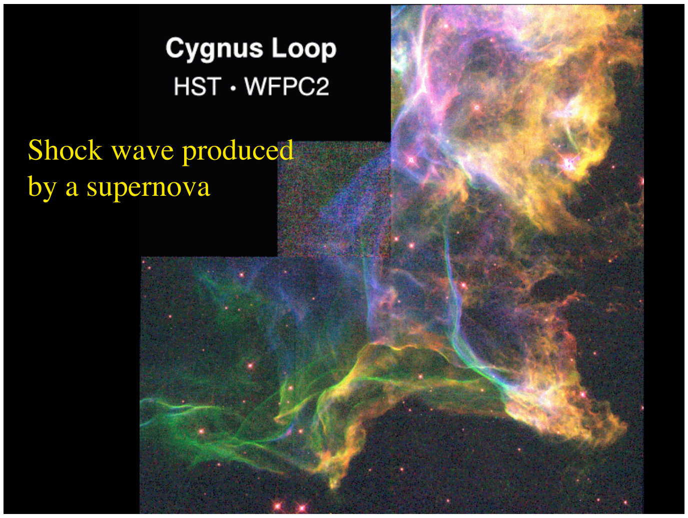
However, discontinuous changes in all variables are possible - ***shocks.***

Physically, shocks result from nonlinear steepening of smooth waves, or from disturbances that propagate faster then the compressive wave speeds in the fluid.

Mathematically, they are exact nonlinear solutions to the PDEs, and solutions that represent discontinuous compression (div(v)<0) and decompression (div(v)>0) are allowed. The latter (rarefaction shocks) are physically impossible since they violate entropy conditions.

Shocks are described by jump conditions, the change in conserved variables across the discontinuity.





Hydrodynamic shocks

In the frame of reference of the shock, there is a steady fluid flow toward the shock from the upstream direction, and away from the shock in the downstream direction.

v_u = shock speed

Mass, momentum and energy must all be conserved in this frame

$$\rho_d v_d = \rho_u v_u$$

$$\rho_d v_d^2 + P_d = \rho_u v_u^2 + P_u$$

$$\frac{1}{2}\rho_d v_d^3 + (e_d + P_d)v_d = \frac{1}{2}\rho_u v_u^3 + (e_u + P_u)v_u$$

Can solve these equations for the Rankine-Hugoniot jump conditions (ratios of downstream to upstream quantities). For example:

$$\frac{\rho_d}{\rho_u} = \frac{(\gamma+1)\mathcal{M}^2}{(\gamma-1)\mathcal{M}^2+2}$$

$M=v_u/C_u$ = shock Mach number

Given the shock Mach number, and quantities in the upstream gas, can use the RK relations to compute properties of the downstream flow.

Some important properties of hydrodynamic shocks:

- Maximum density jump for M >> 1 is $(\gamma+1)/(\gamma-1)$.
- Transverse component of velocity is unchanged through shock
- The downstream flow is always subsonic, $v_d/C_d < 1$. So sound waves in downstream medium can reach shock.

MHD shocks

Can derive the MHD shock jump conditions by applying conservation of mass, momentum, and energy across shock front, in reference frame moving with shock. Also require the normal component of B be continuous, and $\nabla \times (\mathbf{v} \times \mathbf{B})$ be continuous.

v_u = shock speed

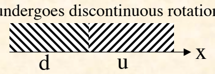
Get many more equations, since there are many more variables in which to specify jump.

Tedious to write general jump conditions. Instead, focus on special cases.

MHD shocks

Alfven "shocks" (rotational discontinuity).

$V_{x,u} = V_{x,d}$ (no compression), but transverse components of V and B change discontunusously (field undergoes discontinuous rotation)



Slow shock

Field "decompresses" through shock, $M_s < C_f$. Field direction deflected towards normal.

Fast shock

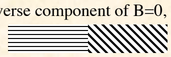
Field compresses through shock. Field direction deflected away from normal.

Switch-on shock

Upstream transverse component of B=0, downstream non-zero

Switch-off shock

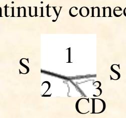
Downstream transverse component of B=0, upstream non-zero



Shocks in multidimensions.

Isolated, planar shocks are not that complicated. However, the interaction of shocks in multidimensions produces a wide variety of complex flows.

Triple points: produced by interaction of two shocks. Since the postshock conditions (2 and 3) can be different, there must be a contact discontinuity connected to the point of interaction.

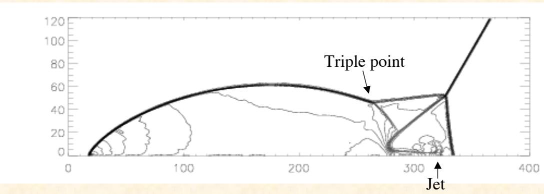


Baroclinic generation of vorticity: Strong shear can be present across the contact discontinuity produced in shock interactions (slip surface). This represents a sheet of vorticity $\omega = \nabla \times \mathbf{v}$. Generated in shock interactions. (Vorticity is also produced in curved shocks.)

Double Mach reflection.

Oblique reflection of a planar shock from a wall.

Movie of density



Linear Instabilities

Going beyond the study of waves and shocks in fluids requires learning about the zoo of MHD instabilities in fluids.

See monographs by Chandrasekhar 1965
Drazin & Reid 1981

Probably the most important are:

1. Gravitational instability.
2. Thermal instability.
3. Rayleigh-Taylor (RT) instability.
4. Richtmyer-Meshkov (RM) instability.
5. Kelvin-Helmholtz (KH) instability.
6. Magneto-rotational instability (MRI)

Why talk about them here? In general, the purpose of numerical methods is to study the nonlinear evolution of these (and other) instabilities. We must be sure our methods capture them *correctly*.

But what does *correctly* mean?

The dispersion relation for most MHD instabilities indicates a very broad (sometimes infinite) range of wavenumbers (wavelengths) are unstable.

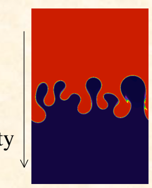
But grid codes can only resolve wavelengths between $N\Delta x$ and L, where N>2 is an algorithm dependent number.

Correctly means the linear growth rates of all modes between $N\Delta x$ and L are represented accurately.

Of course, we want to make N as small as possible. The truncation error in *all* grid codes is not Galilean invariant. So if you are studying instabilities in which fastest growing modes are near N, will get different answers in different frames.

But then you would be studying unresolved flow that is dominated by truncation error. Would ALSO get different answer with different resolutions.

One example: Rayleigh-Taylor instability



Classic instability of a heavy fluid accelerated by a light fluid. Really a buoyancy instability, like convection.

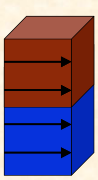
Stability (WKB) analysis is complicated! Only state result:

When growth rate $n^2 = gk\left(\frac{\rho_h - \rho_l}{\rho_h + \rho_l} - \frac{(\mathbf{B}\cdot\mathbf{k})^2}{2\pi g k(\rho_h + \rho_l)}\right) > 0$ INSTABILITY

For $\rho_h > \rho_l$ and B=0, all k are unstable, and highest k have largest growth rate.

Nonlinear evolution of RTI

Stone & Gardiner, Phys. Fluids 2007; Stone & Gardiner, ApJ, 2007

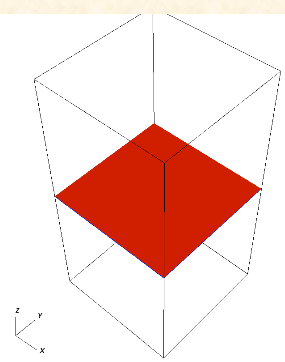


Computational domain *L* x *L* x *2L*,
256 x 256 x 512 grid cells
Uniform vertical acceleration *g*

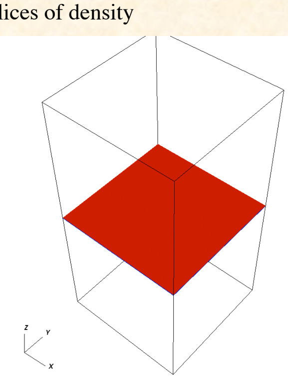
Atwood number $A = \frac{\rho_{\rm Heavy} - \rho_{\rm Light}}{\rho_{\rm Heavy} + \rho_{\rm Light}} = \frac{1}{2}$

Modes unstable parallel to B for $\lambda > B^2/[(\rho_h - \rho_l)g]$
Modes unstable perpendicular to B for all λ

Strong field B = 0.4 B_c (but still very high β);
random perturbations
Isosurface and slices of density



Hydro



MHD

Waves, shocks, and instabilities…

- are all fundamental aspect of fluid dynamics, and must be understood to understand applications.
- Will see in future lectures they make excellent code *tests*.
- Instabilities often lead to turbulence - the study of which is an important code *application*

Numerical Methods.

How do we solve the equations of MHD on a computer?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} + P^*] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = 0$$

$$\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

First step is ***discretization***. We use grid-based methods:

1. Discretize space into (N_x,N_y,N_z) zones: x → (x_i, y_j, z_k)
2. Discretize time into discrete levels: t → t^n
3. Represent dependent variables as *either*
   Point-wise values: $a^n_{i,j,k} = a(x_i, y_j, z_k, t^n)$ a=(ρ,ρv,E)
   Volume averages: $a^n_{i,j,k} = \int_V a(x,y,z,t^n)dV/V$

Difference formulae depend on this choice!

Round-off error

Not all numbers on real axis can be represented.

If floating point operations result in a number that cannot be represented, some sort of rounding must be used.

Rounding is *correct* if no machine number lies between x and its rounded value x'. Difference between x and x' is the *round-off error*.

Can be rigorously proved that the relative error of a rounded value is bounded by a small, machine dependent number (the *machine precision*), that is

$$\frac{|x - x'|}{|x|} < \epsilon$$

Basis for all rigorous error analyses of numerical methods

Truncation error

Numerical algorithms approximate the true (analytic) solution using algebraic operations.

Difference between true and approximate (numerical) solution is ***truncation error***.

TE is not related to the finite precision of numbers on a computer (round-off error). Would exist even on a perfect machine with no round-off error.

TE is under programmers control. Much of numerical analysis is trying to reduce it.

Convergence, consistency, and stability

Consistency: truncation error must decrease as resolution increased.

Convergence: numerical solution should approach analytic solution as grid spacing Δx decreases (numerical resolution increases).

Higher order schemes converge faster, so they are better, right?

Yes, but what matters is the *absolute error*. If the error coefficient in a higher order scheme is large, it may have worse error than a lower order scheme with a smaller coefficient.

$C_1 h$ may be better than $C_2 h^2$ if $C_1 << C_2$

Also, (1) cost and code complexity put a practical limit on how high one should extend high-order schemes, (2) all methods are first-order for discontinuities, and global error may be dominated by shocks and not smooth flow.

Stability: round-off error must remain small and bounded.

Differentiation: finite-differences

Obvious approximation to a derivative is a difference:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Leading term in truncation error is O(h), so "first-order accurate" (think of Taylor series expansion).

In addition to truncation error, there will be round-off error in evaluating derivative, from

- when x >> h, x+h inaccurate
- when evaluating *f(x)*, error is magnified.

Round-off error of simplest form for derivative is *at best* sqrt(ε)

Centered difference

A significantly more accurate estimate for the derivative comes from the centered difference formula:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

In this case, the leading term dropped in the Taylor series expansion is $O(h^2)$! Can lead to order of magnitude decrease in truncation error. Can get a "second-order" scheme almost for free.

Using Taylor series, it is also easy to build successively higher-order approximations for derivative (SMOOTH FUNCTIONS ONLY), e.g. PENCIL code.

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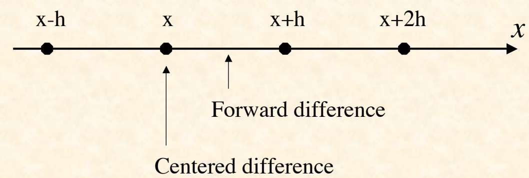
Centering of differences



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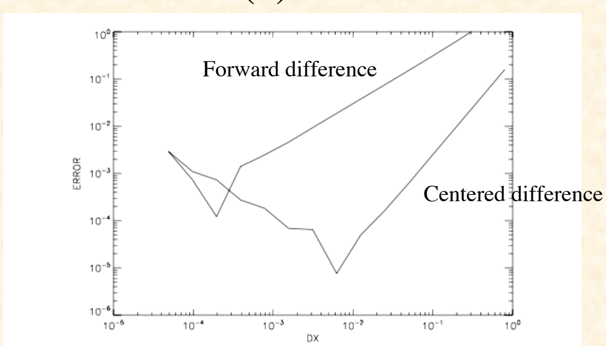
Example: error in derivative for f(x) = x³



Slope of lines is order of method

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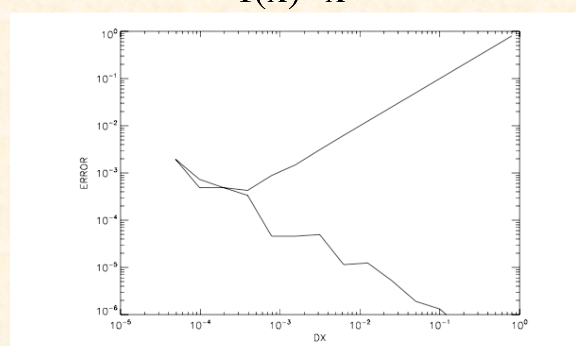
Example: error in derivative for f(x)=x²



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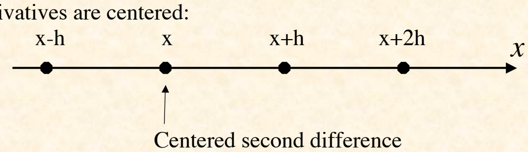
Higher-order derivatives

It is easy to build up difference representations of higher-order derivatives, e.g.

$$\begin{aligned} f''(x) &\approx \left(\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h} \right) / h \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \end{aligned}$$

Accurate to $O(h^2)$ provided h = constant. Always think how derivatives are centered:



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Finite differencing

Simplest discretization of the simplest hyperbolic PDE (scalar linear advection equation) is forward-time centered-space (FTCS):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad \text{becomes} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) = 0$$

Perform von Neumann stability analysis.
For constant coefficients, the *analytic* solution to the difference equation must be of the form: $u_j^n = \xi^n \exp ikj\Delta x$

Substitute this form into FDE, find $\xi(k) = 1 - i\left(\frac{a\Delta t}{\Delta x}\right)\sin k\Delta x$

Note that $|\xi(k)| > 1$ for ***all*** $\Delta t > 0$

Method is *unconditionally unstable*.

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Lax-Friedrichs

Change time derivative in FTCS to use average of u at t^n, get LF:

$$\frac{u_j^{n+1} - (u_{j+1}^n + u_{j-1}^n)/2}{\Delta t} + a\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) = 0$$

von Neumann stability analysis now gives: $\xi(k) = \cos k\Delta x - i\frac{a\Delta t}{\Delta x}\sin k\Delta x$

This has $|\xi(k)| < 1$ iff $\boxed{\frac{a\Delta t}{\Delta x} \le 1}$

This is the *Courant-Levy-Friedrichs (CFL) stability criterion.*

Why does LF work and FTCS does not? Rewrite difference equation:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) + \frac{1}{2}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta t}\right)$$

This is just FD form for the mixed PDE: $\frac{\partial u}{\partial t} + a\left(\frac{\partial u}{\partial x}\right) = \kappa\frac{\partial^2 u}{\partial x^2}$ $\quad \kappa = \frac{\Delta x^2}{2\Delta t}$

LF adds explicit viscosity which makes algorithm stable.

Upwind methods

Write spatial derivative using one-sided differences that depend on sign of velocity

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) \quad \text{if a > 0}$$

$$-a\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) \quad \text{if a < 0}$$

Remarkably, can rewrite upwind FDE in same form as LF:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) + \frac{a}{2\Delta x}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

So upwind method also adds explicit diffusion, with $\kappa = \frac{a\Delta x}{2}$

Define the "CFL number" $C = \frac{\Delta t}{\Delta x / a}$

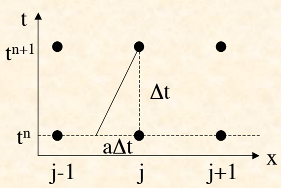
which is the ratio of timestep to maximum stable timestep.

For C < 1, upwind methods add *less* diffusion than LF.

Lax Wendroff

Can interpret FDE using characteristic tracing. Solution to linear advection equation is just initial condition displaced by $a\Delta t$:

$u(x,t^0+\Delta t) = u(x-a\Delta t, t^0)$



(1) Linear interpolation at t^n gives first-order upwind

(2) Quadratic polynomial fit at t^n gives LW

LW finite-difference formulae (compare to upwind and LF):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) + \frac{a^2\Delta t}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

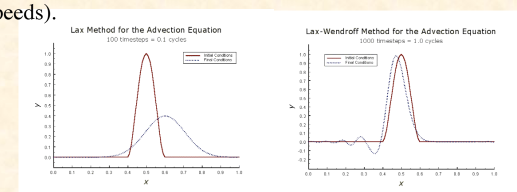
Diffusion versus dispersion error

Analytic solution to FDE may approximate a PDE that is different from the one we are trying to solve (the *modified equation*).

e.g. we saw LF FDE actually solves: $\frac{\partial u}{\partial t} + a\left(\frac{\partial u}{\partial x}\right) = \kappa\frac{\partial^2 u}{\partial x^2}$

Clearly, LF (and first-order upwind) FDE add *diffusion error*.

Can show that LW adds *dispersion error* (different k propagate at different speeds).



Dispersion error can be a serious problem for discontinuous solutions.

Nonlinear terms.

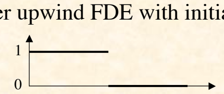
Consider the simplest nonlinear advection equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

(Burger's equation)

Finite-differencing can give completely wrong solutions. For example, consider first-order upwind FDE with initial data:

$u^0 = 1 \quad i < 0$

$\quad\; 0 \quad i > 0$



Use $\frac{u_j^{n+1} - u_j^n}{\Delta t} = -u_j^n\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right)$ RHS always zero! Solution never evolves!

Instead, need to solve nonlinear term in conservative form:

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad \text{where flux} \quad f = u^2/2$$

That is: $u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\left(f_{j+1/2} - f_{j-1/2}\right)$ More about computing f in future lectures.

Summary

- Need to understand *some* MHD to understand numerical methods (and applications!)
  - Linear waves
  - Shocks
  - Instabilities

  They are the basis of excellent code tests.
- Discretization is key to grid-based methods
  - *Finite difference* versus *finite element* versus *finite volume*
- Nonlinear terms and discontinuities pose special challenges.