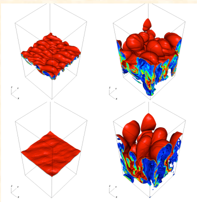
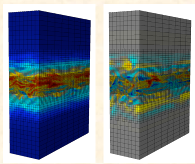
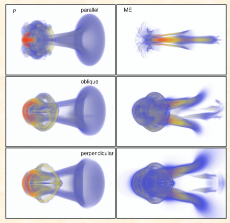
Grid-based methods for hydrodynamics, MHD, and radiation hydrodynamics.

(Four lectures)







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http://www.astro.princeton.edu/~jstone/downloads/papers/Lecture4.pdf

Outline of lectures

Lecture 1. Introduction to physics and numerics

Lecture 2. Operator split (ZEUS-like) methods

Lecture 3. Godunov (PPM-like) methods

Lecture 4. Radiation Hydrodynamics

Lecture 4:
Radiation Hydrodynamics.

1. Additional Physics in grid codes.
   - Optically-thin cooling
   - Microscopic diffusion (viscosity, resistivity, conduction)
   - Gravity
   - Special relativity
2. Radiation hydrodynamics.
3. Numerical methods for radiation hydrodynamics.
   - Full transport methods
   - Flux-limited diffusion
4. Radiation hydrodynamics in Godunov schemes.
5. Future of grid-based methods.

Adding more physics.

For operator split codes, it is self-consistent to add more physics using operator splitting:

1. Update advection ("transport") terms
2. Update source terms
3. Update additional physics terms

For Godunov methods, operator splitting:

1. formally makes scheme first-order in time
2. can lead to stability problems

Nonetheless, operator splitting is still the most commonly used approach to adding new physics in Godunov methods.

Doing Better: source terms in van Leer unsplit integrator.

Adding new physics to van Leer unsplit integrator at second-order is straightforward

Red=changes to algorithm with new physics terms.

Steps in algorithm

1. Compute first-order fluxes at every interface
2. Use these fluxes to advance solution for Δt/2 (predict step), including new physics source terms
3. Use predicted solution to compute new source terms at $t^{n+1/2}$
4. Compute L/R states using time-advanced state, and compute fluxes
5. Advance solution over full time step (correct step) using new fluxes, including new physics source terms computed in step 3

Doing Better: source terms in CTU unsplit integrator

Red=changes to algorithm with new physics terms.

Steps in algorithm:

1. Compute L/R states including time advance using characteristic tracing and source terms for multi-dimensional MHD, and new physics source terms.
2. Compute fluxes from Riemann solver
3. Compute solution at $t^{n+1/2}$
4. Correct L/R states with transverse flux gradients for Δt/2 including source terms for MHD, and new physics source terms, e.g. in 2D x-face states corrected via:

$$\mathbf{q}^{n+1/2}_{L,i-1/2,j} = \mathbf{q}_{L,i-1/2,j} + \frac{\delta t}{2\delta y}\left(\mathbf{g}^*_{i-1,j+1/2} - \mathbf{g}^*_{i-1,j-1/2}\right) + \frac{\delta t}{2}\mathbf{s}_{x,i-1,j}$$
$$\mathbf{q}^{n+1/2}_{R,i-1/2,j} = \mathbf{q}_{R,i-1/2,j} + \frac{\delta t}{2\delta y}\left(\mathbf{g}^*_{i,j+1/2} - \mathbf{g}^*_{i,j-1/2}\right) + \frac{\delta t}{2}\mathbf{s}_{x,i,j}$$

5. Compute multi-dimensional fluxes from corrected L/R states
6. Advance solution full time step using multi-dimensional fluxes, and new source terms computed solution at $t^{n+1/2}$ from step 3

Optically-thin cooling

Adds source terms to energy equation: $\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F}_E = -\rho^2 \Lambda(T) + \rho H$

Where Λ(T) is per-particle cooling rate, H is per particle heating rate.

Depending on cooling function, terms are usually nonlinear in E, and very stiff. *Forward Euler* differencing requires very small Δt

Better to use *Backward Euler* (fully implicit) differencing, where source terms are calculated at advanced time (using E^{n+1}).

Not difficult to add cooling directly to integrator in Godunov methods.

Warning: easy to add cooling, but makes physics of MHD much more complex. For example, need to add thermal conduction to be able to resolve Field length to get correct dynamics with cooling instability.

Moral: Don't add physics just because you can. It takes work to really understand what is going on in both the physics and numerics.

Viscosity and thermal conduction

Momentum equation: $\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F}_{\rho \mathbf{v}} = \nabla \cdot (\nu \nabla \mathbf{v})$

Energy equation: $\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F}_E = \nabla \cdot (\kappa \nabla \mathbf{T}) + \nabla \cdot (\mathbf{v} \cdot (\nu \nabla \mathbf{v}))$

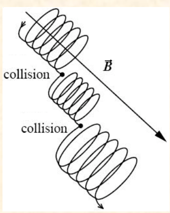
Both cases can be differenced using FTCS: $\frac{q_j^{n+1} - q_j^n}{\Delta t} = D\left(\frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{\Delta t}\right)$

But stability constraint on FTCS for parabolic equations is very restrictive $\Delta t_D \leq (\Delta x)^2/4D$

Solution: (1) sub-stepping: take many steps at Δt_D for every MHD Δt.
(2) super-timestepping; size of sub-time steps varied.
(3) implicit differencing: $\frac{q_j^{n+1} - q_j^n}{\Delta t} = D\left(\frac{q_{j+1}^{n+1} - 2q_j^{n+1} + q_{j-1}^{n+1}}{\Delta t}\right)$

Latter leads to large sparse-banded matrices in 2D and 3D, which must be solved using, e.g. multigrid.

Anisotropic conduction and resistivity



In a magnetized, weakly collisional plasma the thermal conduction and viscous transport will be mainly along field lines. Produces *qualitative* change in the dynamics (magneto-thermal instability, heat-flux buoyancy instability, magneto-viscous instability).

Study through the inclusion of anisotropic viscous and heat fluxes.

$\mathbf{Q} = -\chi \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla T$ Anisotropic heat flux (χ = conductivity)

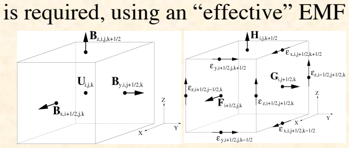
$\mathbf{\Pi} = -3\eta\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathsf{I}\right)\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathsf{I}\right) : \nabla \mathbf{v}$ Anisotropic viscous stress tensor

Difference using FTCS with monotonic transverse temperature or velocity gradients (Parrish & Stone 2005; Sharma & Hammett 2007)
Can represent 1:1000 anisotropies in flux with any orientation of **B** on grid.

Resistivity

Induction equation becomes: $\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}) = 0$
J = current density

Now over-riding concern is keeping div(B)=0. This suggests a CT differencing is required, using an "effective" EMF **E**=η**J** located at cell corners



Once again, time step constraint very restrictive: $\Delta t_\eta \leq (\Delta x)^2/4\eta$

Can use (1) sub-stepping, or (2) super-timestepping. Implicit CT differencing is complex.

Can be extended to ambipolar-diffusion and Hall regimes by appropriate definition of diffusive EMF.

Gravity

With gravity, momentum and energy equations can be written as:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F}_{\rho \mathbf{v}} + \nabla \cdot \mathbf{G} = 0 \qquad \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F}_E = \rho \mathbf{v} \cdot \mathbf{g}$$

Where **g**=gravitational accn, $\mathbf{G} = (\mathbf{gg} - g^2/2)/4\pi G$ gravitational stress tensor

For fixed gravitational potential (e.g. central star)
- Momentum is not conserved
- Total energy is conserved

So add source term to momentum equations using analytic form for acceleration, and add source term to total energy using mass fluxes and potential difference - conserves total energy exactly

For self gravity
Add source terms to momentum as divergence of gravitational stress tensor - conserves total momentum exactly. Add source terms to total energy using mass fluxes and potential difference.

Of course, must also solve Poisson's equation for the potential: use time average of Φ^n and Φ^{n+1} to ensure second order accuracy without solving PE twice per timestep.

Special relativity

Such a substantial change to algorithm that it can be considered as writing a new solver rather than extending existing solver.

SR MHD equations can also be written in conservative form $\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0$

But definition of conserved variables (and their fluxes) is more complicated:

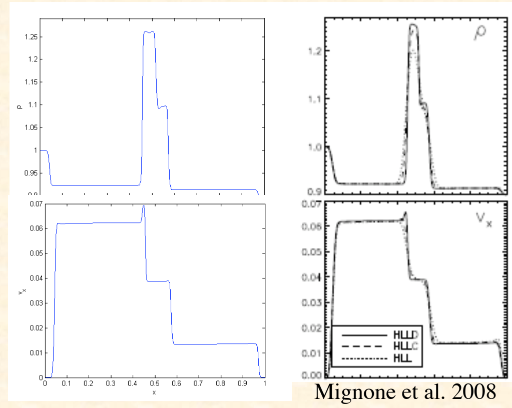
$$\mathbf{U} = f(\mathbf{W}) = \begin{pmatrix} \gamma\rho \\ (w+b^2)\gamma^2 v_x - b_0 b_x \\ (w+b^2)\gamma^2 v_y - b_0 b_y \\ (w+b^2)\gamma^2 v_z - b_0 b_z \\ (w+b^2)\gamma^2 - P_t - b_0^2 \\ B_y \\ B_z \end{pmatrix}.$$

$b_0 = \gamma(\mathbf{B} \cdot \mathbf{v})$
$\mathbf{b} = \mathbf{B}/\gamma + \gamma(\mathbf{B} \cdot \mathbf{v})\mathbf{v}$
$w = \rho + [\Gamma/(\Gamma - 1)]P$ (enthalpy)

So overall integration algorithm remains the same
1. Reconstruction step
2. Compute fluxes with Riemann solver
3. van Leer unsplit integrator

But significant changes required in each step:
1. Conversion from conserved to primitive variables requires nonlinear root finding, we use method of Noble et al. (2006)
2. Relativistic Riemann solver required (HLL, HLLC, HLLD)
3. Use van Leer unsplit integrator since no characteristic decomposition needed in reconstruction step.

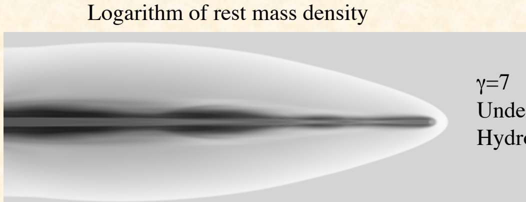
Relativistic MHD shock test



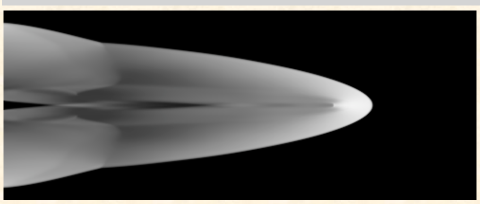
Mignone et al. 2008

2D tests: relativistic slab jets

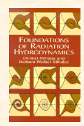
Logarithm of rest mass density

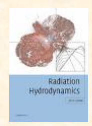


γ=7
Underdense
Hydrodynamic jet



γ=3.2
Underdense
MHD jet





Foundations of Radiation Hydrodynamics

Numerical MHD is easy compared to radiation hydrodynamics.

Some of the reasons why radiation MHD is hard:
- Which frame (co-moving, mixed-frame, fully relativistic)?
- Proper closure of moment equations. Variable Eddington factor is expensive, flux-limited diffusion of questionable accuracy.
- Mathematical problem changes in different regimes: *hyperbolic* in streaming limit, mixed *hyperbolic-parabolic* in diffusion limit.
- Wide range of timescales requires semi-implicit methods.
- Frequency dependent transport.
- Non-LTE effects.

This complexity means that radiation hydrodynamics means different things to different people.

In some cases, only need to include energy transport via material-radiation energy exchange term:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = -g^0$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

Optically thin cooling.
Heating by (ionizing) radiation.

Examples: diffuse ISM, HII regions.

In some cases, may need to include energy transport by diffusion (in optically thick regions) as well as material-radiation energy exchange term:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = -g^0$$

$$\frac{\partial E_r}{\partial t} + \nabla \cdot D \nabla E_r = g^0$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

Examples: dense ISM, protostellar disks

In some cases, may "only" need to include momentum exchange terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = -\mathbf{g}$$

$$\mathbf{g} = \frac{1}{c}\int d\nu \int_{4\pi} d\Omega \mathbf{n}(j_\nu - \kappa_\nu k_\nu)$$

e.g. line-driven winds (assuming gas is isothermal).

Of course, computing **g** can be extremely difficult!

In some cases, need to include *both* energy *and* momentum exchange terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = -\mathbf{g}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = -g^0$$

$$\frac{\partial E_r}{\partial t} + \nabla \cdot \mathbf{F}_r = g^0$$

$$\frac{1}{c^2}\frac{\partial \mathbf{F}_r}{\partial t} + \nabla \cdot \mathsf{P}_r = \mathbf{g}$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

$$\mathbf{g} = \frac{1}{c}\int d\nu \int_{4\pi} d\Omega \mathbf{n}(j_\nu - \kappa_\nu k_\nu)$$

Examples:
radiation-dominated disks
core-collapse SN

All of these problems could be called "radiation hydrodynamics".

Obviously, the numerical methods required in each regime are very different.

Grid-based method versus particles for radiation transfer

Even though we use a grid for the MHD, we could still choose to use either a grid or particles (Monte Carlo) to solve the transfer equation.

Grid:
- More accurate and less noise
- Ideally suited for GPUs
- Difficult to extend to include scattering, and line-transport
- Very expensive

Particles:
- Very flexible, easy to extend to frequency-dependent transport, etc.
- Embarassingly parallel
- Noisy, especially in optically thick regions

Will only discuss grid approach here.

Moment equation approach.

Fundamental description of the radiation is the frequency-dependent transfer equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \nabla \cdot (\mathbf{n} I_\nu) = j_\nu - \kappa_\nu I_\nu$$

Can be thought of as the "collisionless Boltzmann equation for photons", so that *I* is the "photon distribution function".

So, just like the fluid equations, can take moments over phase space (angles) and frequency to derive a set of moment equations.

Why? Reduces dimensions of problem, making it easier to solve.

Radiation Hydrodynamics in ZEUS-2D

Stone & Mihalas 1992, Stone, Mihalas, & Norman 1992

- Solve comoving moment equations to O(1) in (v/c)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{c}\chi_F \mathbf{F}$$

$$\rho \frac{D}{Dt}\left(\frac{E}{\rho}\right) = -\nabla \cdot \mathbf{F} - \nabla \mathbf{v} : \mathsf{P} + 4\pi\kappa_P B - c\kappa_E E$$

$$\rho \frac{D}{Dt}\left(\frac{e+E}{\rho}\right) = -\nabla \mathbf{v} : \mathsf{P} - P\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}$$

$$\frac{\rho}{c^2}\frac{D}{Dt}\left(\frac{\mathbf{F}}{\rho}\right) = -\nabla \cdot \mathsf{P} - \frac{1}{c}\chi_F \mathbf{F}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- Use variable Eddington tensor $\mathsf{f} = \mathsf{P}/E$ to close hierarchy.
- Compute f from formal solution using short characteristics (Mihalas, Auer, & Mihalas 1978, Kunasz & Auer 1988).

Solve with an operator-split approach.

Use implicit solution of moment equations in "source step"

$$\frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{F} - \nabla \mathbf{v} : \mathsf{P} + 4\pi\kappa_P B - c\kappa_E E$$

$$\frac{\partial (e+E)}{\partial t} = -\nabla \mathbf{v} : \mathsf{P} - p\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}$$

$$\frac{1}{c^2}\frac{\partial \mathbf{F}}{\partial t} = -\nabla \cdot \mathsf{P} - \frac{1}{c}\chi_F \mathbf{F},$$

Use conservative update of advection terms in "transport step"

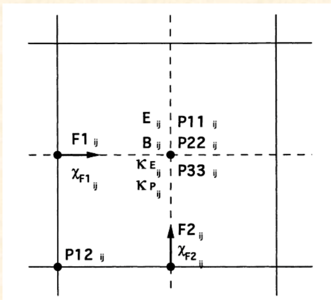
$$\frac{d}{dt}\int_V E dV = -\oint_{dV} E(\mathbf{v} - \mathbf{v_g}) \cdot \mathbf{dS}$$

$$\frac{d}{dt}\int_V (e+E) dV = -\oint_{dV} (e+E)(\mathbf{v} - \mathbf{v_g}) \cdot \mathbf{dS},$$

$$\frac{d}{dt}\int_V \mathbf{F} dV = -\oint_{dV} \mathbf{F}(\mathbf{v} - \mathbf{v_g}) \cdot \mathbf{dS}$$

Centering of variables.

Uses a staggered grid, with scalars at cell-centers, vectors at faces, and off-diagonal tensor components at corners.



Implicit differencing.

Material-radiation interaction and radiation transport terms have a very restrictive time step limit, and must be solved implicitly. Write schematically as

$$E_{i,j}^{n+1} - E_{i,j}^{n} - \Delta t\left[4\pi\kappa P_{i,j}^{n+\theta} B_{i,j}^{n+\theta} - c\kappa E_{i,j}^{n+\theta} E_{i,j}^{n+\theta} - (\nabla\cdot\mathbf{F}_{i,j}^{n+\theta} - (\nabla\mathbf{v}:\mathbf{P}^{n+\theta})_{i,j}\right] = 0,$$

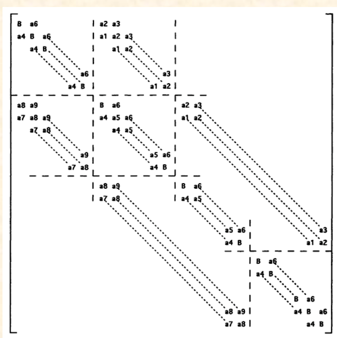
$$e_{i,j}^{n+1} + E_{i,j}^{n+1} - e_{i,j}^{n} - E_{i,j}^{n} - \Delta t\left[-(\nabla\cdot\mathbf{F}^{n+\theta})_{i,j} - (\nabla\mathbf{v}:\mathbf{P}^{n+\theta})_{i,j} - p_{i,j}^{n+\theta}(\nabla\cdot\mathbf{v})_{i,j}\right) = 0.$$

θ=1 is backward Euler (fully implicit)

θ=1/2 is Crank-Nicholson

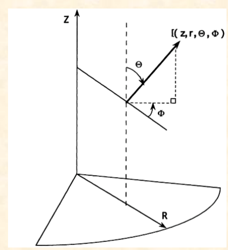
Equations are nonlinear in unknowns, so must use Newton-Raphson iteration to solve. Requires solving large sparse-banded matrix for *every* NR iteration.

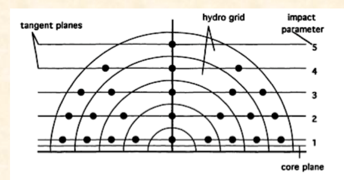
Linear solvers



Matrix solved for each NR iteration is very sparse, so use iterative methods like GMRES or ICCG.

Formal solution to compute Eddington factor restricted to cylindrical coordinates.



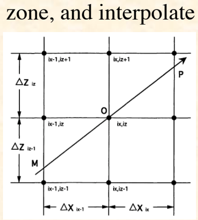


Have to solve the transfer equation along rays $\frac{\partial I}{\partial s} = \chi(S - I)$

to compute the Eddington tensor $\mathsf{f} = \frac{\mathsf{P}}{E} = \frac{\oint I\mathbf{n}\mathbf{n}d\omega}{\oint I d\omega}$,

Short versus long characteristics

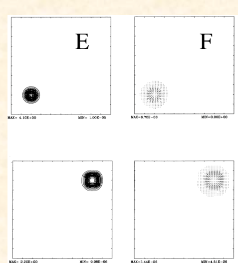
Short characteristics (Kunasz & Auer 1988): solve along ray segments that cross a single zone, and interpolate *I* to start of next ray segment, O(N^2) in 2D



Long characteristics: for each cell, solve along rays that cross entire grid, O(N^3) in 2D.

Short characteristics are much faster, but more diffusive.

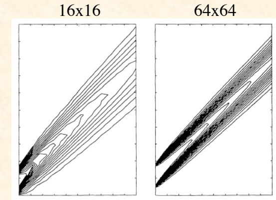
Module tested extensively.



Dynamic diffusion test: advection of Gaussian pulse of radiation.

16x16

64x64



Contours of I in searchlight beam test of formal solution.

But the module was never used for a published application!

Radiation Hydrodynamics in 3D with ZEUS

Turner & Stone 2001

Studying accretion disk turbulence requires 3D. In 1998, formal solution via short characteristics in 3D was not feasible.

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{c}\chi_F \mathbf{F}$$

$$\rho \frac{D}{Dt}\left(\frac{E}{\rho}\right) = -\nabla \cdot \mathbf{F} - \nabla \mathbf{v} : \mathsf{P} + 4\pi\kappa_P B - c\kappa_E E$$

$$\rho \frac{D}{Dt}\left(\frac{e+E}{\rho}\right) = -\nabla \mathbf{v} : \mathsf{P} - P\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}$$

Solution:
Close moment equations using flux-limited diffusion. →

$$\mathbf{F} = -\frac{c\lambda}{\chi}\nabla E$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Solve with an operator-split approach.

Use implicit solution of moment equations in "source step"

$$\frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{F} - \nabla \mathbf{v} : \mathsf{P} + 4\pi\kappa_P B - c\kappa_E E$$

$$\frac{\partial (e+E)}{\partial t} = -\nabla \mathbf{v} : \mathsf{P} - p\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}$$

Use conservative update of advection terms in "transport step"

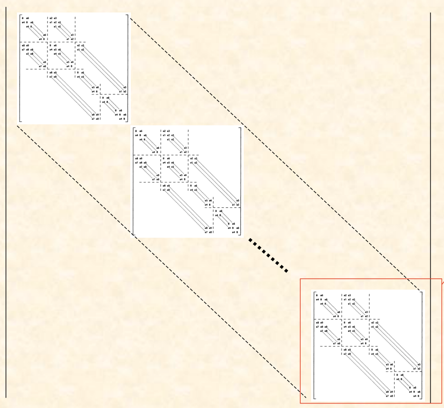
$$\frac{d}{dt}\int_V E dV = -\oint_{dV} E(\mathbf{v} - \mathbf{v_g}) \cdot \mathbf{dS}$$

$$\frac{d}{dt}\int_V (e+E) dV = -\oint_{dV} (e+E)(\mathbf{v} - \mathbf{v_g}) \cdot \mathbf{dS},$$

No flux equation with FLD.

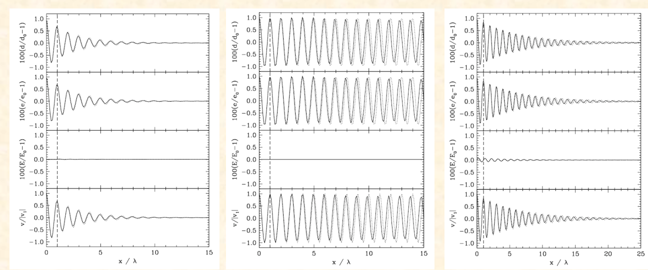
Implicit methods in 3D

Matrix to be solved in *each* NR step is **much** bigger in 3D.



Equivalent to entire matrix in 2D problem

Test: damping of linear waves.

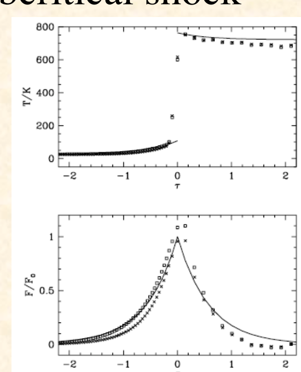


$\tau/\lambda = 0.001$ $\tau/\lambda = 1$ $\tau/\lambda = 1000$

$P_{rad}/P_{gas}=0.1$

Test: subcritical shock

X = Minerbo limiter
□ = Levermore & Pomraning limiter



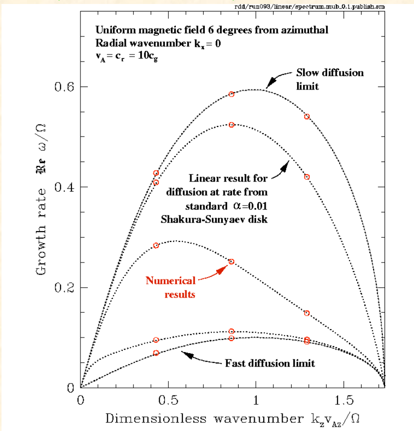
Parameters same as in Sincell, Gehmeyr & Mihalas (1999)

This module was used for local radiation MHD models of the inner regions of accretion disks.

Motivation:

1. What is the saturation amplitude of the MRI in a radiation-dominated plasma?
2. Need to include radiation to balance heating for truly steady-state disk models → spectra.

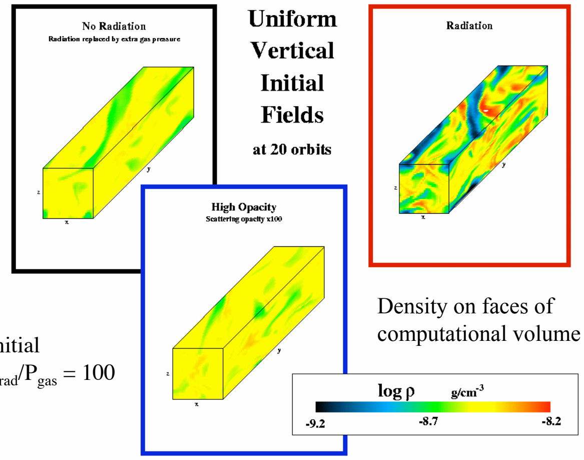
Linear growth rates of the MRI are changed by radiative diffusion (Blaes & Socrates 2001)



(Turner, Stone, & Sano 2002)

Uniform Vertical Initial Fields

at 20 orbits



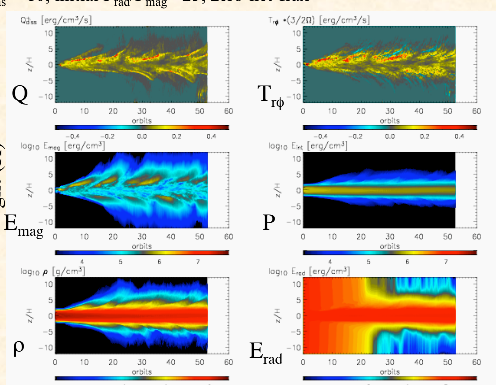
Initial $P_{rad}/P_{gas} = 100$

Density on faces of computational volume

Vertically stratified radiation dominated disks

- Grid is $2H \times 4H \times 24H$ (32x64x384)
- $P_{rad}/P_{gas} = 10$, initial $P_{rad}/P_{mag} = 25$, zero-net-flux

Hirose, Krolik, & Stone 2005



Vertical height (H)

Time (orbits)

Extending Athena to radiation MHD: mixed-frame to O(v/c)

(with Mike Sekora at Princeton)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\mathbf{m}) = 0$$

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla\cdot\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) + \nabla p = -P\mathbf{S_F} = -P\left(-\sigma_t\left(\mathbf{F_r} - \frac{\mathbf{v}E_r + \mathbf{v}\cdot\mathbf{P_r}}{C}\right) + \sigma_a\frac{\mathbf{v}}{C}\left(T^4 - E_r\right)\right)$$

$$\frac{\partial E}{\partial t} + \nabla\cdot\left((E+p)\frac{\mathbf{m}}{\rho}\right) = -PCS_E = -PC\left(\sigma_a\left(T^4 - E_r\right) + \left(\sigma_a - \sigma_s\right)\frac{\mathbf{v}}{C}\cdot\left(\mathbf{F_r} - \frac{\mathbf{v}E_r + \mathbf{v}\cdot\mathbf{P_r}}{C}\right)\right)$$

$$\frac{\partial E_r}{\partial t} + C\nabla\cdot\mathbf{F_r} = CS_E = C\left(\sigma_a\left(T^4 - E_r\right) + \left(\sigma_a - \sigma_s\right)\frac{\mathbf{v}}{C}\cdot\left(\mathbf{F_r} - \frac{\mathbf{v}E_r + \mathbf{v}\cdot\mathbf{P_r}}{C}\right)\right)$$

$$\frac{\partial \mathbf{F_r}}{\partial t} + C\nabla\cdot\mathbf{P_r} = C\mathbf{S_F} = C\left(-\sigma_t\left(\mathbf{F_r} - \frac{\mathbf{v}E_r + \mathbf{v}\cdot\mathbf{P_r}}{C}\right) + \sigma_a\frac{\mathbf{v}}{C}\left(T^4 - E_r\right)\right)$$

$$\mathbf{P_r} = \mathsf{f}E_r, \qquad C = \frac{c}{a_\infty}, \qquad P = \frac{a_r T_\infty^4}{\rho_\infty a_\infty^2}, \qquad a_r = \frac{8\pi^5 k^4}{15 c^3 h^3}$$

Mihalas & Klein 1982, Lowrie & Morel 1999; 2001

Algorithmic Steps – Hybrid Godunov

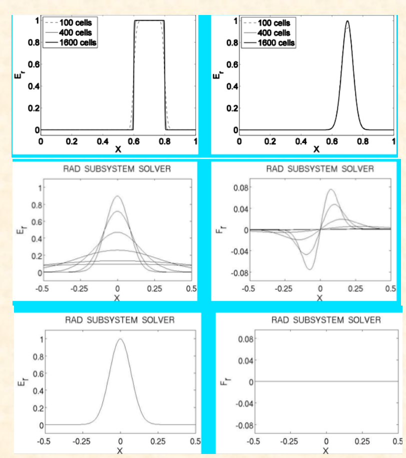
1. **Backward Euler HLLE** (implicitly advances radiation quantities – 1st order). Equations solved in this step are linear.
2. **Modified Godunov Predictor** (couple source effects to hyperbolic structure – 2nd order)
3. **Modified Radiation Riemann Solver** (HLLE, LF, LW, etc.)
4. **Modified Godunov Corrector** (semi-implicitly advances material quantities – 2nd order)

This algorithm works in all asymptotic limits…

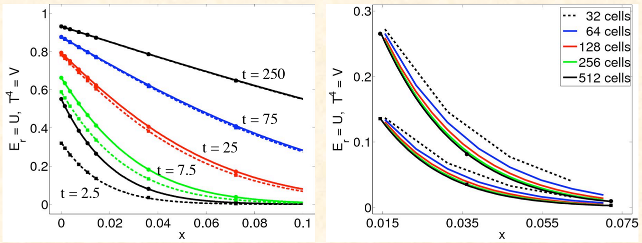
Streaming-limit
τ << 1
Pulse propagates at c

Static diffusion limit
v/c << λ/l
Gaussian pulse diffuses

Dynamic diffusion limit
v/c ~ λ/l
Gaussian pulse advected with fluid



Marshak waves



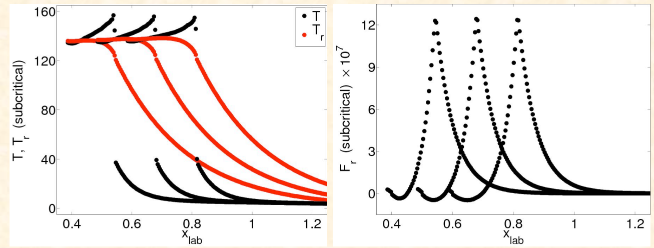
——— = E_r

.... = T

Points are semi-analytic solution of Su & Olson (1996).

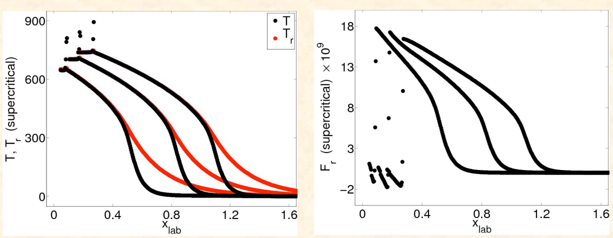
Sub-critical shock

Profiles at three different times.



Super-critical shock

Profiles at three different times.



Parameter values as in Ensman (1994), Turner & Stone (2001), Hayes & Norman (2003)

Convergence rate of linear waves.

(testing underway)

Will test whether our splitting of an implicit solution of the radiation moment equations from modified Godunov solution of the material equations is 2nd order in all (any?) limits.

Overall summary of 4 lectures.

What we covered:

- Basic MHD processes
- Operator split methods (like ZEUS)
- Godunov methods (like Athena)
- Extension with more physics
- Radiation Hydrodynamics (a whole new regime for codes)

What we left out:

- Other schemes (central schemes, WENO, spectral methods,…)
- Other codes (Pencil, RAMSES, HARM, Cosmos++, …)
- **Applications!** (90% of my papers are applications)

Future of Athena

1. Cylindrical Grid (developed by A. Skinner UMd)
2. Special Relativistic MHD (being tested)
3. Dust Particles (being tested)
4. Parallelized AMR (Dec. 2009)
5. Full Transport radiation MHD (??)

Watch the Athena web pages for more information. New version with SMR should be released within the year (by 1/1/2010). New Trac site will be made open to public this year.

Future of grid-based methods.

Some things are easy to predict:

- More physics
- Better methods
- Better resolution on bigger machines

But the real future is with YOU, the students of this program.

Thank you for your attention and feedback.