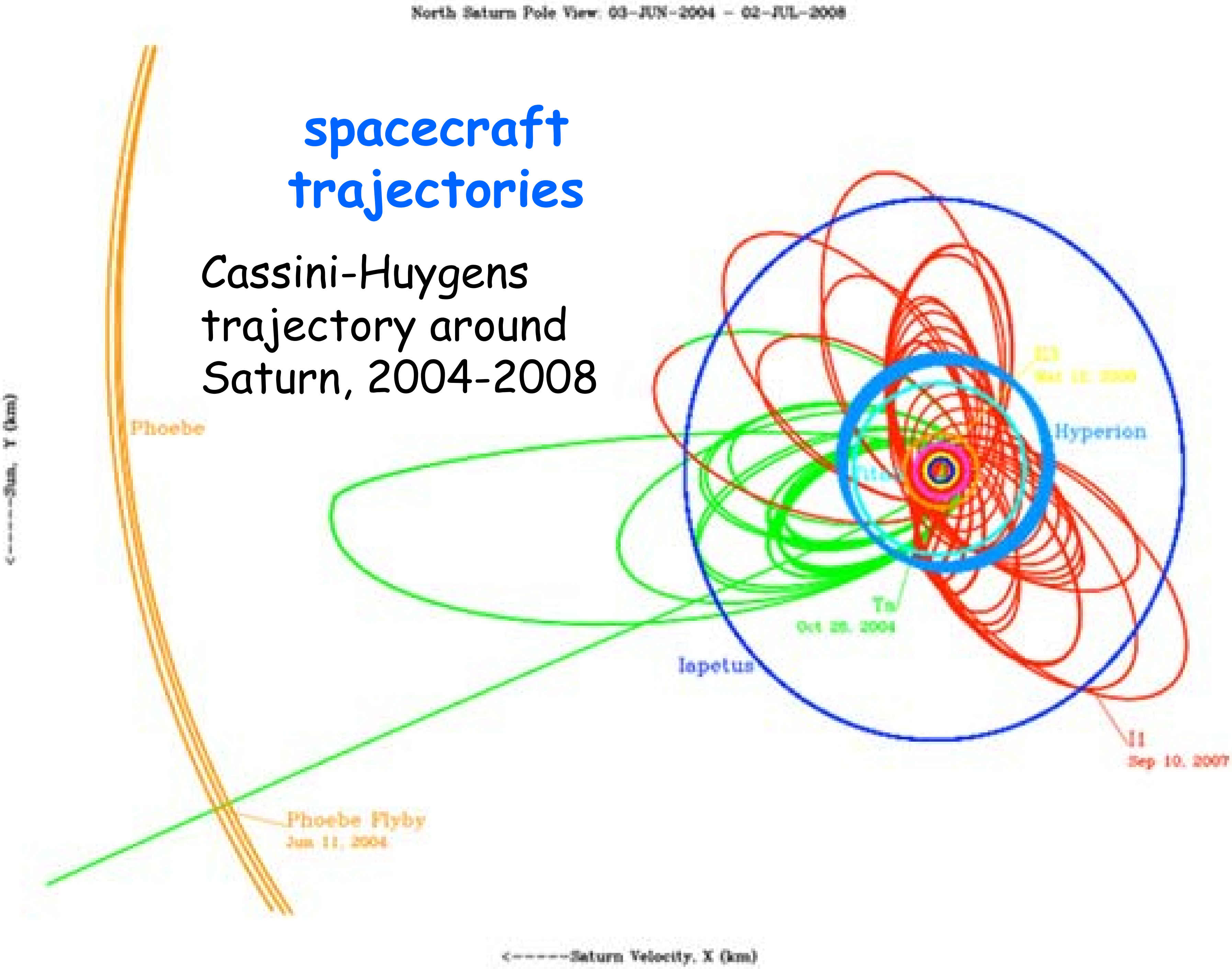
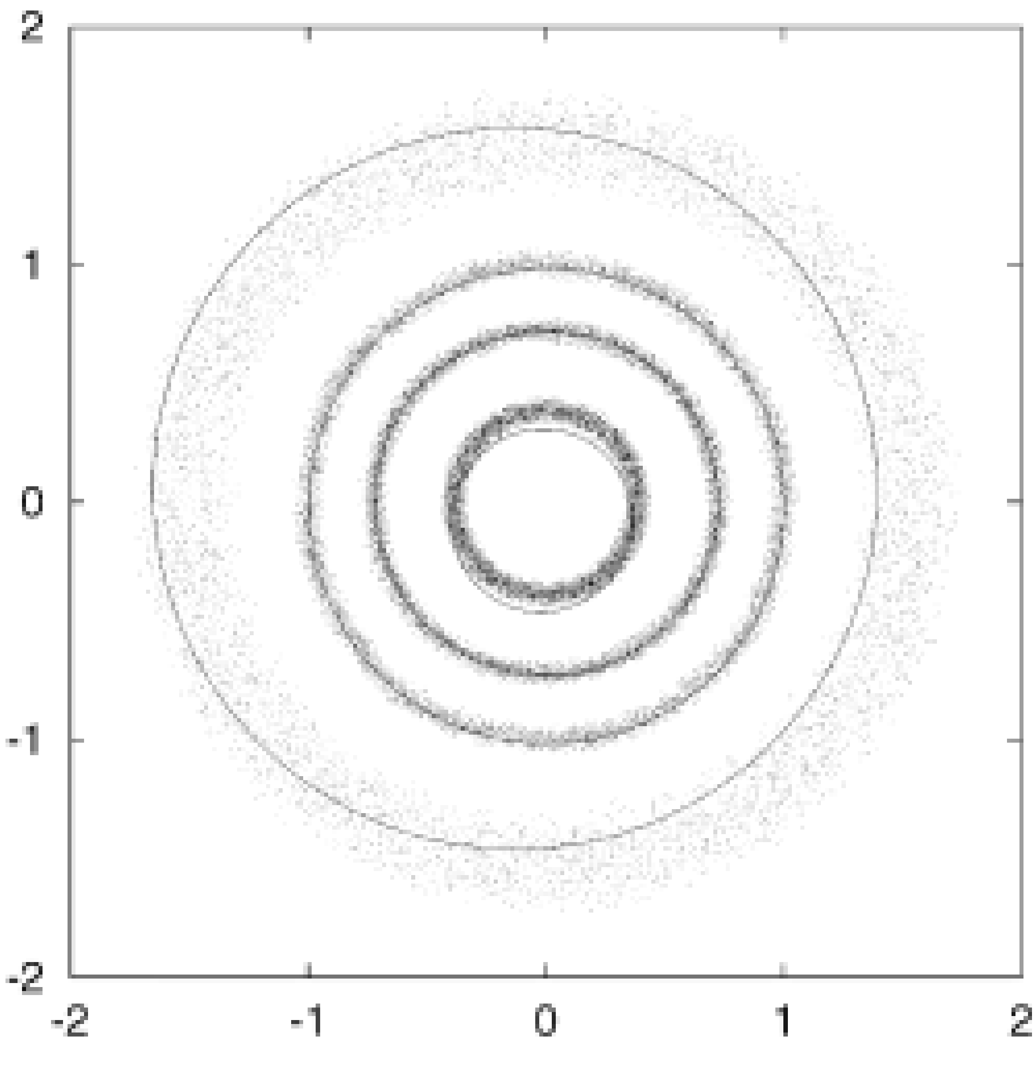
# Geometric methods for orbit integration

PiTP 2009 – Scott Tremaine

# spacecraft trajectories

Cassini-Huygens trajectory around Saturn, 2004-2008



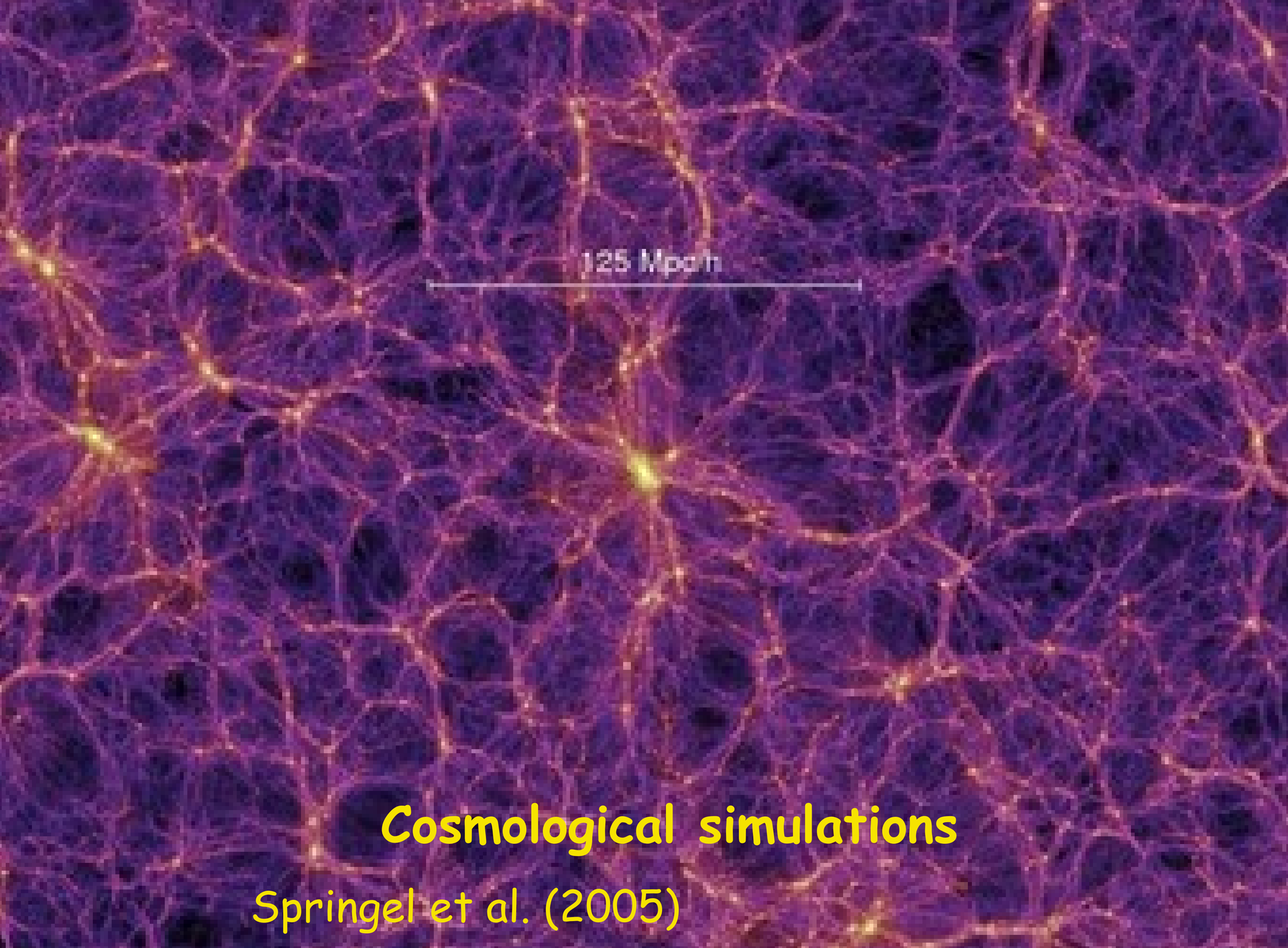



# Planetary orbits

lines = current orbits of the four inner planets

dots = orbits of the inner planets over 50,000 years, 4.5 Gyr in the future

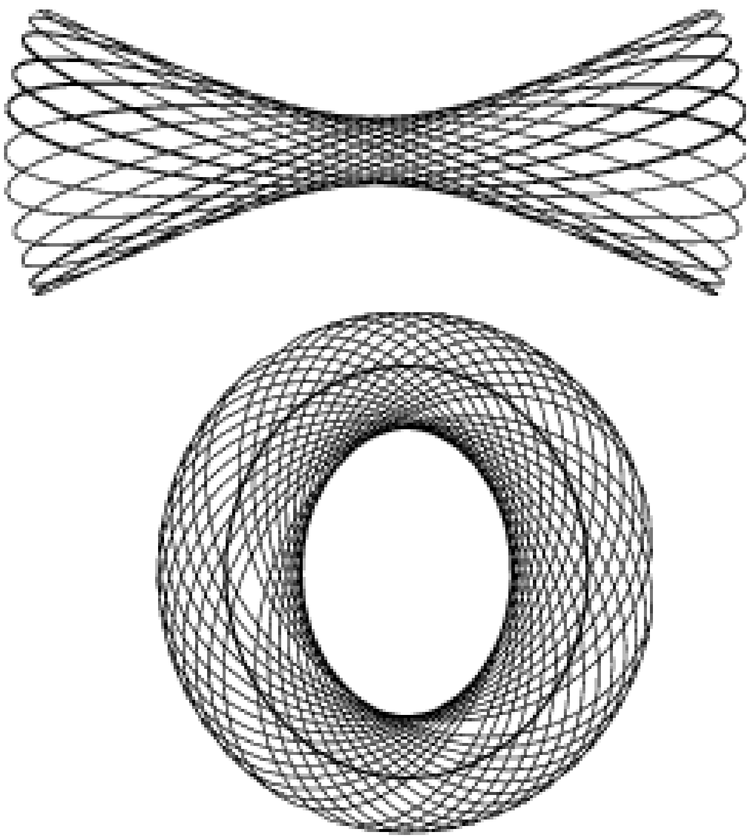
Ito & Tanikawa (2002)



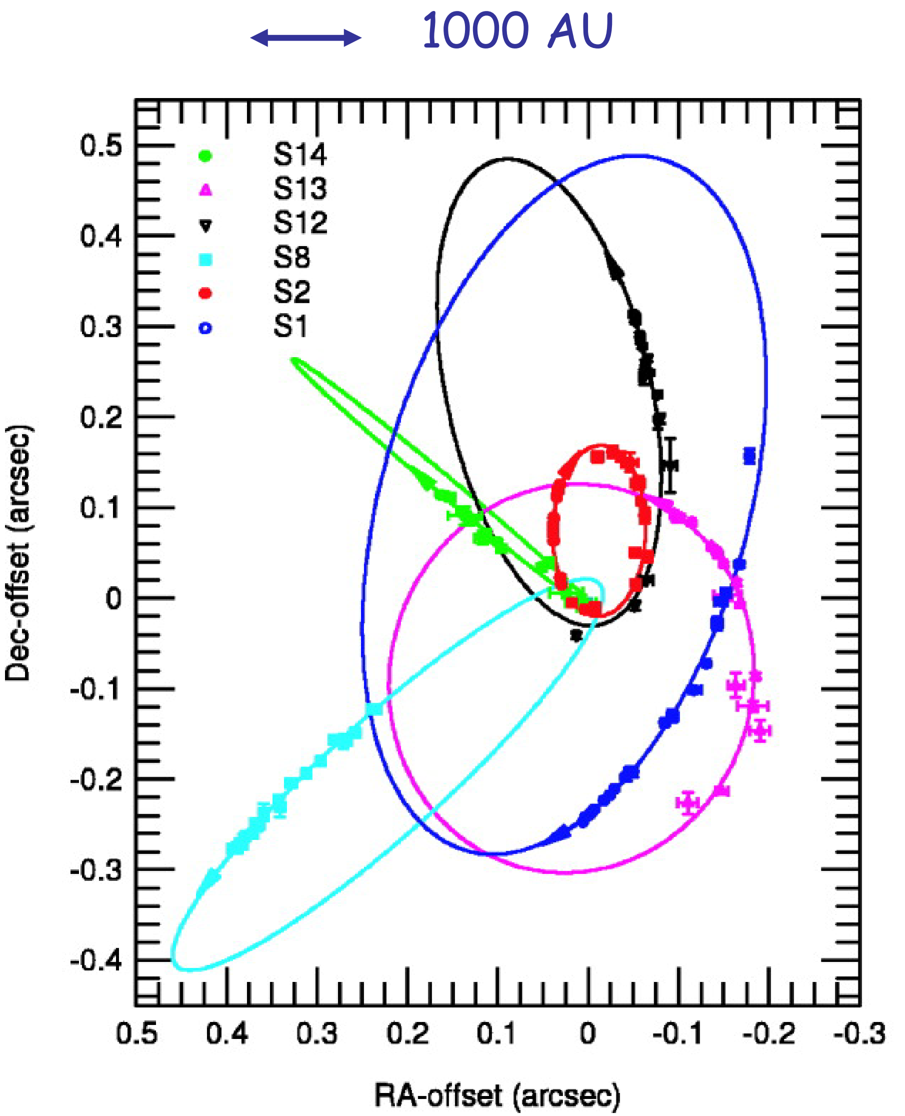

**Cosmological simulations**

Springel et al. (2005)

# Galactic dynamics



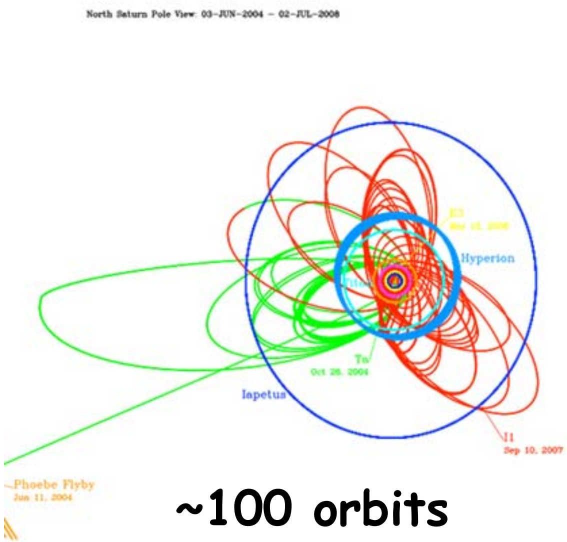
box and tube orbits in a galactic potential



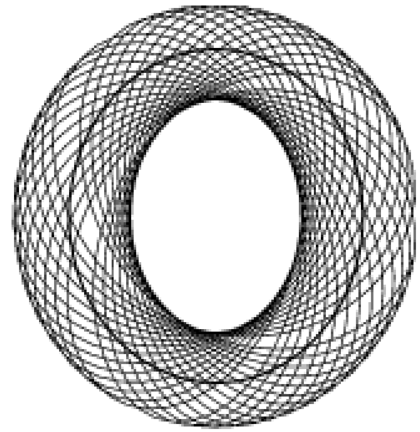

orbits of stars near the Galactic center

Eisenhauer et al. (2005)

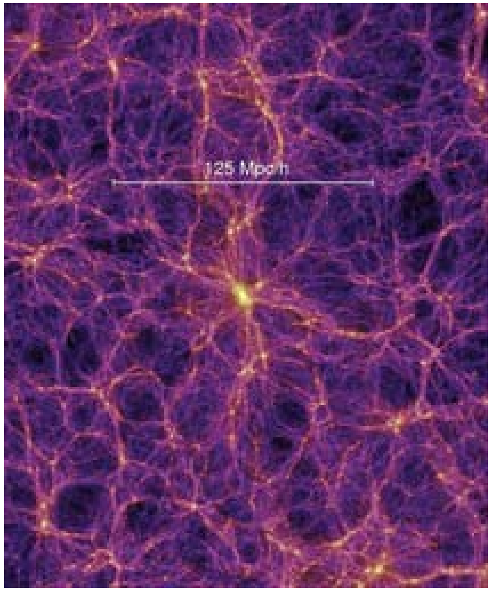

Large Hadron Collider



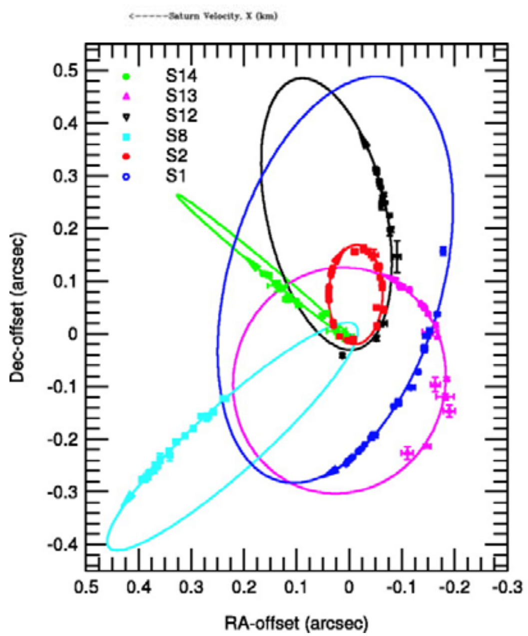

~100 orbits



~100-1000 orbits

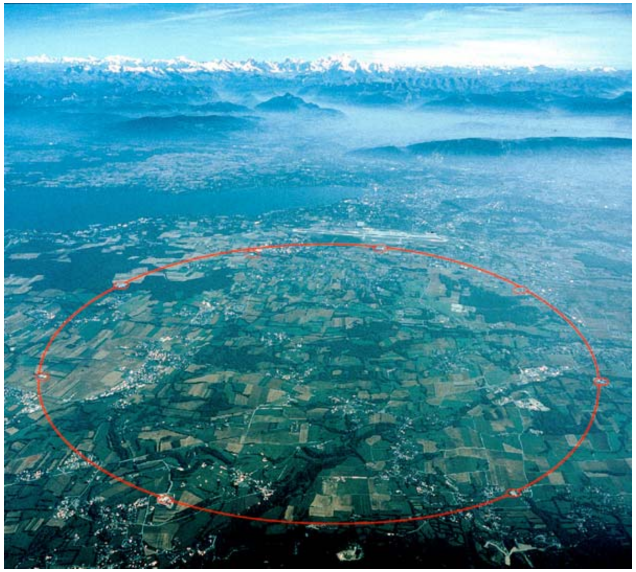


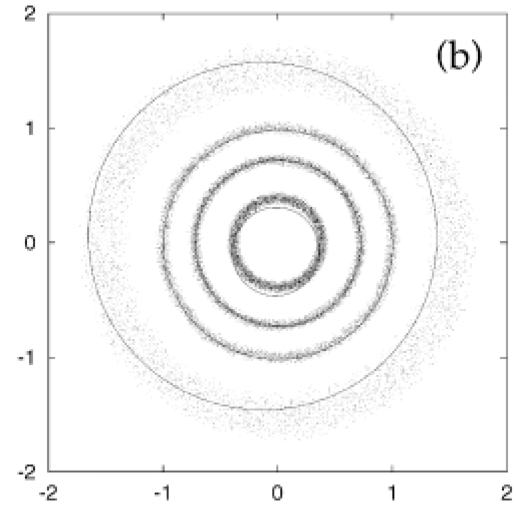

~100-1000 orbits




~$10^6$ orbits

~$10^9$ orbits






~$10^{10}$ orbits

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Consider following a particle in the force field of a point mass. Set G=M=1 for simplicity. Equations of motion read

$$\dot{\mathbf{r}} = \mathbf{v} \quad ; \quad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}) = -\frac{\hat{\mathbf{r}}}{r^2}$$

Examine three integration methods with timestep *h:*

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}_n)$$

1. Euler's method

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}_{n+1})$$

2. modified Euler's

$$\mathbf{r}' = \mathbf{r}_n + \frac{1}{2}h\mathbf{v}_n \ ; \ \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}') \ ; \ \mathbf{r}_{n+1} = \mathbf{r}' + \frac{1}{2}h\mathbf{v}_{n+1}$$

3. leapfrog

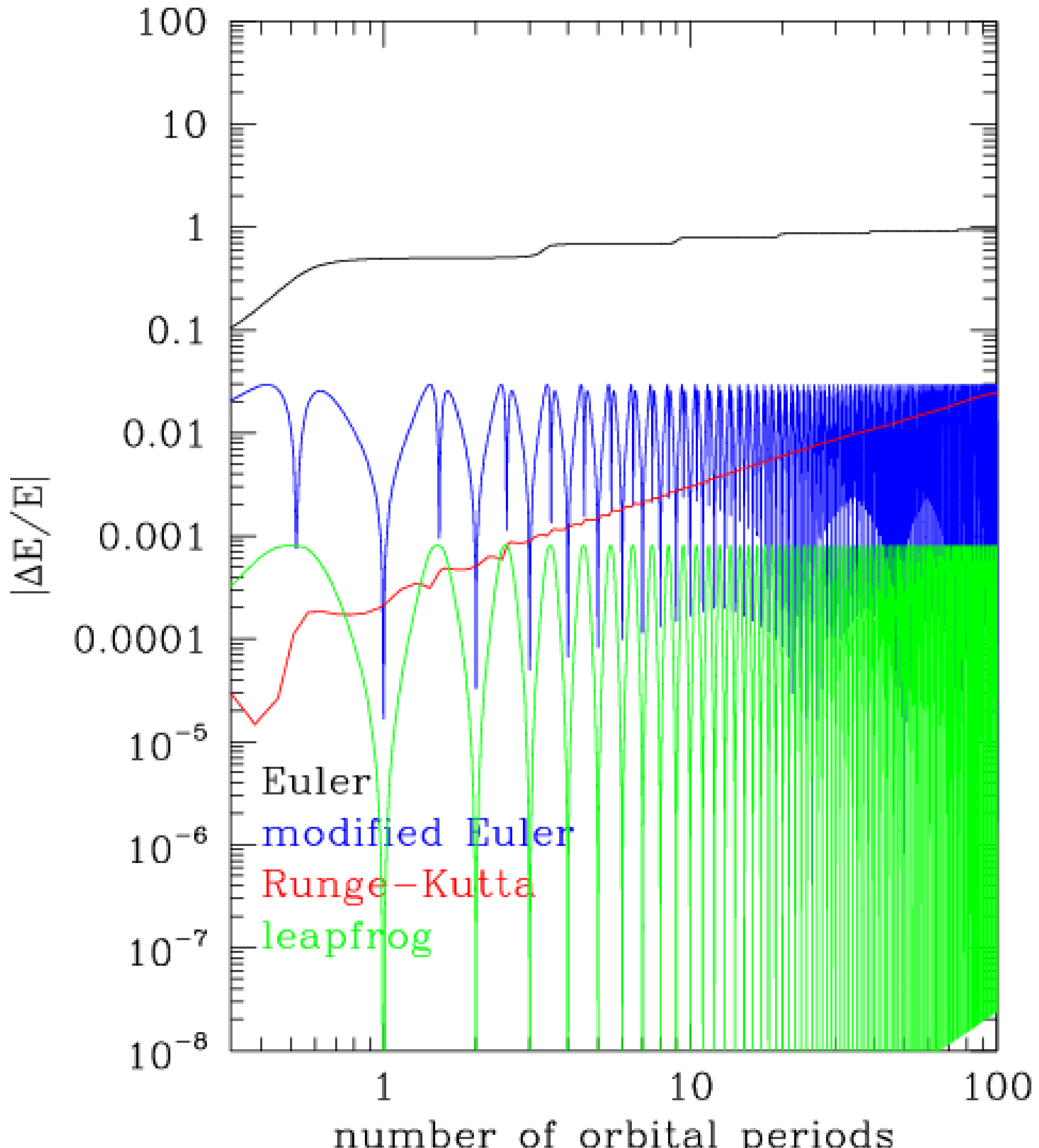
4. Runge-Kutta method

Euler methods are first-order; leapfrog is second-order; Runge-Kutta is fourth order

To keep the playing field level, use equal number of force evaluations per orbit for each method (rather than equal timesteps) eccentricity = 0.2

200 force evaluations per orbit

plot shows fractional energy error $|\Delta E/E|$



# Liouville's theorem

The flow in phase space generated by a dynamical system governed by a Hamiltonian conserves volume

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A **geometric integration algorithm** is a numerical integration algorithm that exactly preserves some geometric property of the original set of differential equations

## Volume-conserving algorithms:

- conserve phase-space volume, i.e. satisfy Liouville's theorem
- appropriate for Hamiltonian systems
- e.g. modified Euler, leapfrog but *not* Runge-Kutta

## Energy-conserving algorithms:

- conserve energy, i.e. restrict the system to a surface of constant energy in phase space
- appropriate for systems with time-independent Hamiltonians, e.g. motion in a fixed potential
- does *not* include modified Euler, leapfrog, Runge-Kutta

## Time-reversible algorithms:

- integrate forward in time for N steps, reverse all velocities, integrate backward in time for N steps, reverse velocities, and the system is back where it started
- appropriate for time-reversible systems, e.g. gravitational N-body problem
- includes leapfrog but not modified Euler or Runge-Kutta

$$\mathbf{r}' = \mathbf{r}_n + \frac{1}{2} h \mathbf{v}_n \ ; \ \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}') \ ; \ \mathbf{r}_{n+1} = \mathbf{r}' + \frac{1}{2} h \mathbf{v}_{n+1}$$

## Symplectic algorithms:

- if the dynamical system is described by a Hamiltonian H(q,p) then

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \quad ; \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$$

- if y(t)=[q(t),p(t)] then the flow from $y(t_0)$ to $y(t_1)$ generated by a Hamiltonian is a **symplectic** or canonical map
- an integration method is symplectic if the formula for advancing by one timestep

$$y_{n+1}=y_n+g(t_n,y_n,h)$$

  is also a symplectic map, i.e. if it can be generated by a Hamiltonian
- for one-dimensional systems symplectic = volume-conserving (actually area-conserving)
- for systems of more than one dimension symplectic is more general
- modified Euler and leapfrog are symplectic

The motivation for geometric integration algorithms is that **preserving the phase-space geometry of the flow determined by the real dynamical system is more important than minimizing the one-step error**

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# Geometric integrators for cosmology

As Volker showed, the Hamiltonian in comoving coordinates is

$$H(\mathbf{q},\mathbf{p},t) = \sum \mathbf{p}_i \cdot \mathbf{q}_i - L = H_A(\mathbf{q},\mathbf{p},t) + H_B(\mathbf{q},\mathbf{p},t)$$

with

$$H_A = \sum_i \frac{\mathbf{p}_i^2}{2m_i a^2(t)}, \qquad H_B = -\sum_{i>j} \frac{G m_i m_j}{a(t)|\mathbf{x}_i - \mathbf{x}_j|}.$$

Drift and kick operators correspond to motion under $H_A$ and $H_B$:

$$\mathbf{x}_i' = \mathbf{x}_i + \frac{\mathbf{p}_i}{m_i} \int_t^{t+h} \frac{dt'}{a^2(t')}, \qquad \mathbf{p}_i' = \mathbf{p}_i - \frac{G m_i m_j (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \int_t^{t+h} \frac{dt'}{a(t')}$$

# Geometric integrators for planetary systems

To follow motion in the general potential $\Phi$(r,t) we may use the Hamiltonian splitting

$$H(\mathbf{q},\mathbf{p},t) = H_A(\mathbf{q},\mathbf{p},t) + H_B(\mathbf{q},\mathbf{p},t)$$

with

$$H_A = \frac{1}{2}p^2, \quad H_B = \Phi(\mathbf{q},t)$$

Motion of a test particle in a planetary system is described by

$$\Phi(\mathbf{r},t) = -\frac{GM_*}{r} - \sum_j \frac{Gm_j}{|\mathbf{r}-\mathbf{r}_j|}$$

In this case a much better split is

$$H_A = \frac{1}{2}p^2 - \frac{GM_\star}{r}, \quad H_B = \sum_j \frac{Gm_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

The workhorse for long orbit integrations in planetary systems is the mixed-variable symplectic integrator (Wisdom & Holman 1991)

$$H(\mathbf{r}, \mathbf{p}) = H_A + H_B,$$

with

$$H_A = \frac{1}{2}p^2 - \frac{GM_\star}{r}, \qquad H_B = \sum_j \frac{Gm_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

- integrate $H_A$ and $H_B$ using leapfrog
- motion under $H_A$ is analytic (Keplerian motion) and motion under $H_B$ is also analytic (impulsive kicks from the planets)
- this is a geometric integrator (symplectic and time-reversible)
- errors smaller than leapfrog by of order $m_{planet}/M_*$ $10^{-4}$
- long-term errors reduced to $O(m_{planet}/M^*)^2$ by techniques such as warmup (start with small timesteps and adiabatically change them)

The workhorse for long orbit integrations in planetary systems is the mixed-variable symplectic (MVS) integrator (Wisdom & Holman 1991)

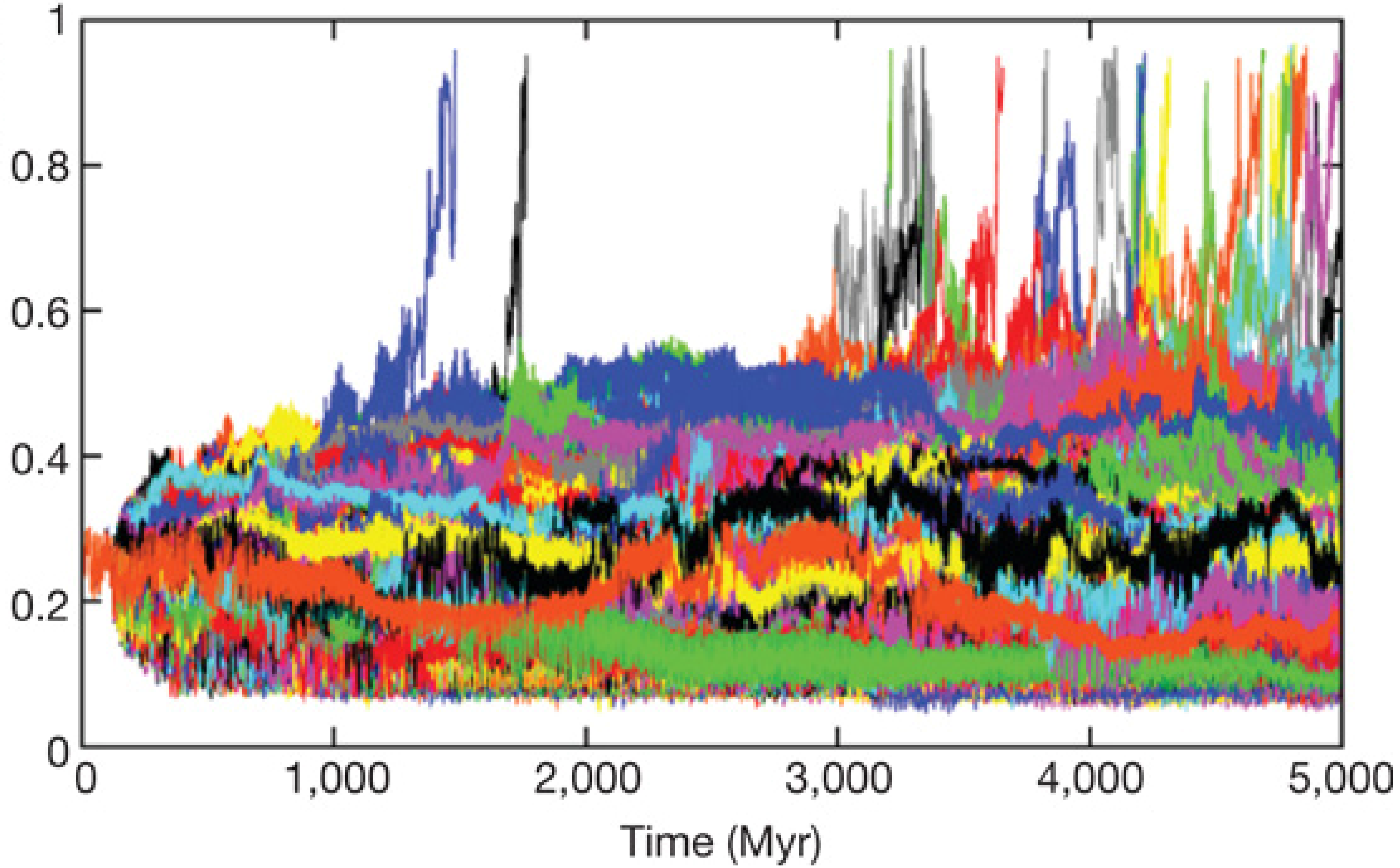
• what it does well: long (up to Gyr) integrations of planets on orbits that are not too far from circular and don't come too close

• what it doesn't do well: close encounters and highly eccentric orbits

The most popular public software packages for solar-system and other planetary integrations are MERCURY (John Chambers) and SWIFT (Hal Levison, Martin Duncan) – URLs are on the wiki

- include several integrators: MVS, Bulirsch-Stoer, Forest-Ruth, etc.
- can handle close encounters + test particles
- can include most important relativistic corrections

Following 9 planets for $10^6$ yr takes about 30 minutes




eccentricity of Mercury over 5 Gyr from 2,500 integrations differing by < 1 mm in semi-major axis of Mercury

(Laskar & Gastineau 2009)

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# Leapfrog with variable timestep (1)

- we want to allow a variable timestep that depends on phase-space position, h= $\tau$(r,v)
- time-reversible integrators have almost all the good properties of symplectic integrators
- define a symmetric function s(h,h'), e.g. s(h,h')=(h+h')/2

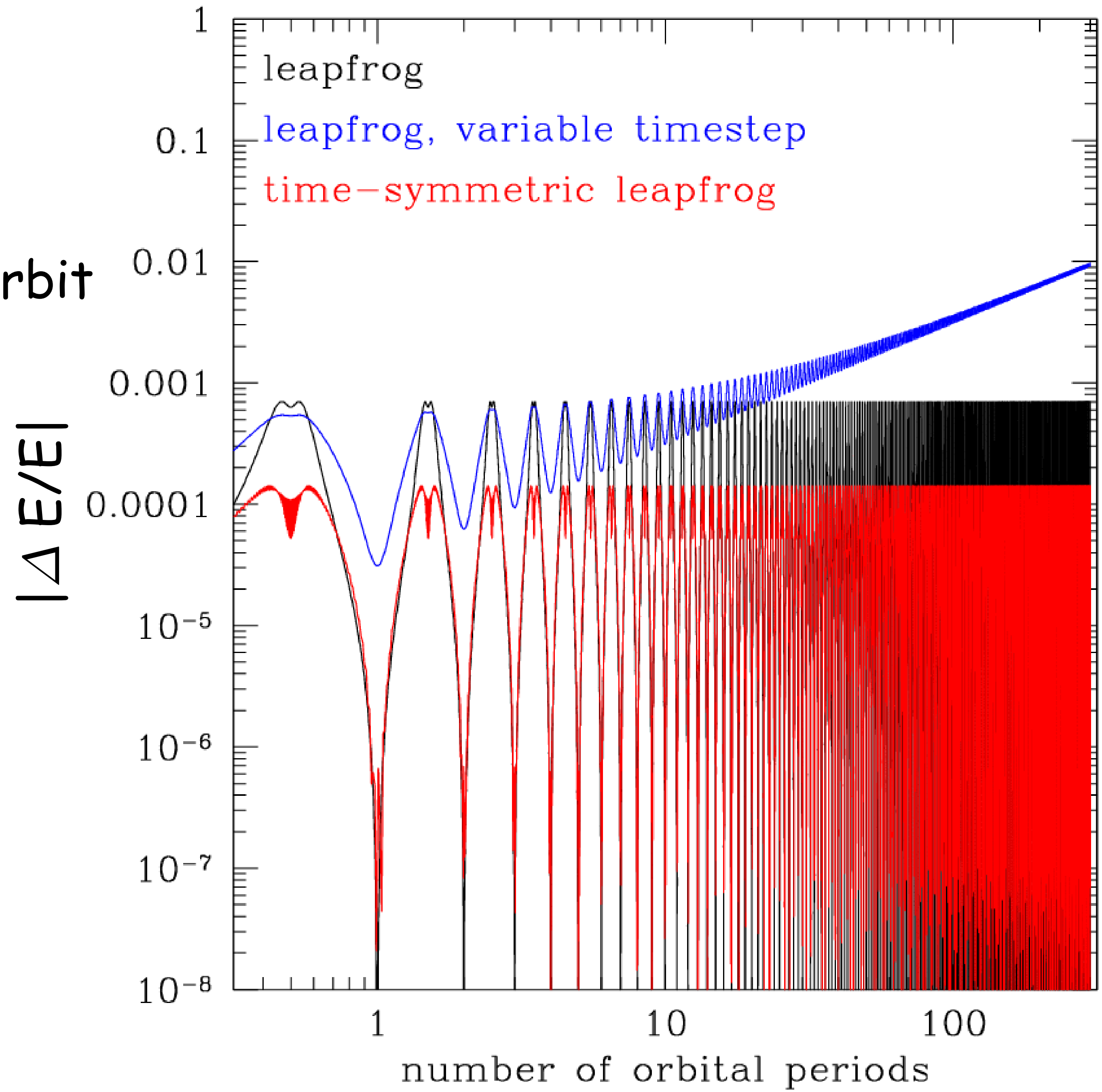
$$\mathbf{r}' = \mathbf{r}_n + \frac{1}{2}h\mathbf{v}_n \quad ; \quad \mathbf{v}' = \mathbf{v}_n + \frac{1}{2}h\mathbf{F}(\mathbf{r}')$$

$$s(h, h') = \tau(\mathbf{r}', \mathbf{v}')$$

$$\mathbf{v}_{n+1} = \mathbf{v}' + \frac{1}{2}h'\mathbf{F}(\mathbf{r}') \quad ; \quad \mathbf{r}_{n+1} = \mathbf{r}' + \frac{1}{2}h'\mathbf{v}_{n+1}$$

This is time-reversible but not symplectic e=0.5

200 steps per orbit



# Leapfrog with variable timestep (2)

**Time transformation:**

- we want to allow a variable timestep that depends on phase-space position $h=\tau(q,p)$
- introduce a new time variable $t'$ by $dt = \tau(q,p)\, dt'$ ; then unit timestep in $t'$ corresponds to desired timestep in $t$
- introduce extended phase space $Q=(q_0,q)$ with $q_0=t$ and $P=(p_0,p)$ with $p_0=-H$. Then set

$$H'(Q,P) = \tau(q,p)[H(q,p)+p_0]$$

If $(q,p)$ satisfy Hamilton's equations with Hamiltonian $H$ and time $t$, then $(Q,P)$ satisfy Hamilton's equations with Hamiltonian $H'$ and time $t'$

- works very well on eccentric orbits but only for one particle (can't synchronize timesteps of different particles)

# Leapfrog with variable timestep (3)

- we have a general differential equation dy/dt = f(t,y) that is known to be time-reversible
- we want an integration scheme that is time-symmetric with a variable timestep that depends on y, h= $\tau$ (y)
- define a symmetric function s(h,h'), e.g. s(h,h')=(h+h')/2
- pick your favorite one-step integrator, $y_{n+1}=y_n+g(y_n,h)$ (e.g. Runge-Kutta)
- introduce a dummy variable z and set $z_n=y_n$ at step n

$$\mathbf{y}' = \mathbf{y}_n + g(\mathbf{z}_n, h/2) \quad ; \quad \mathbf{z}' = \mathbf{z}_n - g(\mathbf{y}', -h/2)$$

$$s(h, h') = \tau(\mathbf{y}')$$

$$\mathbf{z}_{n+1} = \mathbf{z}' + g(\mathbf{y}', h/2) \quad ; \quad \mathbf{y}_{n+1} = \mathbf{y}' - g(\mathbf{z}_{n+1}, -h/2)$$

This is time-reversible (Mikkola & Merritt 2006)

# What has been left out

- individual timesteps
- regularization (Burdet, Kustaanheimo-Stiefel, etc.)
- non-geometric methods for N-body integration (e.g. Hermite methods, multistep and multivalue methods)
- roundoff eror
- homework